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SOLUTION OF THE ABEL INTEGRAL TRANSFORM FOR A CYLINDRICAL LUMINOUS REGION WITH OPTICAL DISTORTIONS AT ITS BOUNDARY

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SOLUTION OF THE ABEL INTEGRAL TRANSFORM FOR A CYLINDRICAL LUMINOUS REGION WITH OPTICAL DISTORTIONS AT ITS BOUNDARY*

Earl R. Mosburg, Jr. and Matthew S. Lojko

The use of orthogonal polynomial expansions in the calculation of the Abel integral transform is discussed. Particular attention is directed to the effects of optical and instrumental distortions when the luminous region is contained by a cylindrical glass tube. An easily calculable solution of the Abel integral is presented which reduces the effect of such distortions by employing a weighting function which has a maximum at the center and vanishes at the boundary. This approach results in a more accurate solution of the Abel integral transform in the case where significant optical and instrumental distortions are present near the boundary of the luminous region.

Key Words: Abel transform, Abel inversion, plasma diagnostics, emissivity profile, radiance profile.

INTRODUCTION

In order to obtain the radial distribution of volume light emissivity within a cylindrical, non-absorbing luminous region, we must solve the Abel integral transform using the projected brightness profile, measured by scanning the detector in a direction perpendicular to the axis of the tube. If the projected brightness profile is $f(x)$, where x is the ratio of the distance off axis to the radius of the luminous region, and if $g(r)$ is the corresponding volume emissivity distribution, where r is the normalized radius, then the Abel integral transforms can be written as

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$$g(r) = -\frac{2}{\pi} \frac{d}{rdr} \int_r^1 \frac{f(x)x\,dx}{\sqrt{x^2-r^2}} \quad (1a)$$

and

$$f(x) = \int_x^1 \frac{g(r)\,r\,dr}{\sqrt{r^2-x^2}} \quad . \quad (1b)$$

or alternatively in the forms

$$g(r) = -\frac{2}{\pi} - \int_r^1 \frac{f'(x)\,dx}{\sqrt{x^2-r^2}} \quad (1c)$$

and

$$f(x) = g(1)\sqrt{1-x^2} - \int_x^1 \sqrt{r^2-x^2}\, g'(r)dr \quad (1d)$$

where the primes indicate differentiation. A direct numerical solution for g(r) using measured values of f(x) in Eq. (1a) or Eq. (1c) is subject to considerable error due to the behavior of the denominator in the integrand and to the necessity for numerical differentiation. These difficulties are considerably alleviated by first making a least square fit of f(x) to a power series expansion as described by Freeman and Katz.[1]

A more convenient expansion in terms of orthogonal polynomials has been reported by Herlitz[2] using Tchebycheff polynomials of the second kind. Popenoe and Shumaker[3] have used Herlitz's method as well as an expansion in terms of Legendre polynomials. The use of orthogonal polynomial expansions is equivalent to a weighted least squares analysis. But here, because of the orthogonality of the basis functions, the coefficients can be independently calculated. Each coefficient can then be tested for statistical significance and the expansion appropriately truncated without any prior, ad hoc decision about the number of terms to be used. The weighting function $w(x)/v(x)$ of Eq. (5) is determined once a particular series of orthogonal polynomials is chosen for the expansion.

Sufficient attention has not, however, been given to the use of orthogonal polynomial expansions in the case where optical or instrumental distortions are introduced at the boundary of the luminous region, as for example, by the presence of a glass container. In this case, distortions due to the scattering and uneven refraction[4] of light in the tube walls may become important. These effects are a maximum near $x = 1$ where a near grazing angle is involved in the measurement. Furthermore, the finite size of the spectrometer slit introduces an averaging over the normalized spacial resolution function of the instrument, $R(x-\zeta)$, such that the measured curve becomes a function, $h(\zeta)$, where

$$h(\zeta) = \int_{\zeta - D/2}^{\zeta + D/2} f(x)R(x-\zeta)dx \qquad (1e)$$

and $\pm D/2$ are the limiting values of $(x-\zeta)$ for which there is appreciable contribution to the integral. The projected brightness profile, $f(x)$, can, in principle, be recovered from $h(\zeta)$ by an appropriate inversion of Eq. (1e), but residual errors will be present. These errors will also be larger near $x = 1$ where the differences between functions $h(\zeta)$ and $f(x)$ are largest, i.e., where the second derivative of $h(\zeta)$ is more important. These distortions are particularly large when it is desired to invert projected profiles approximating $f(x) = \sqrt{1-x^2}$, which corresponds to $g(r) = \text{constant}$. Here the second derivative of $h(\zeta)$ is even larger near the boundary and the luminosity is now high in the region of maximum distortion.

We wish to stress at this point that, in contrast to the distortions, most projected brightness profiles of experimental interest vanish at $x = 1$ and exhibit maxima at or near the center of the light source. It is now clear that in order to reduce the effect of the distortions, we would like a weighting function in Eq. (5) which vanishes at $x = 1$ and exhibits a maximum at $x = 0$.

In this paper we restrict our choice of polynomial to the general class of Ultraspherical or Gegenbauer polynomials, which includes Legendre and Tchebycheff polynomials as special cases. In what follows we will use the notation of the Handbook of Mathematical Functions.[5] We expand $f(x)$ in terms of general Gegenbauer polynomials as

$$f(x) = \sum_{n=0}^{N} a_n v(x) C_n^{(\alpha)}(u(x)) \qquad (2)$$

where $v(x)$ is some shape function to be chosen and $u(x)$ is some function of x. Substituting Eq. (2) into Eq. (1a) we arrive at the expression

$$g(r) = -\frac{2}{\pi} \sum_{n=0}^{N} a_n \frac{d}{rdr} \int_r^1 \frac{v(x) C_n^{(\alpha)}(u(x)) x \, dx}{\sqrt{x^2 - r^2}} . \qquad (3)$$

When the orthonormalization integral for the Gegenbauer polynomials[5] is written in the form

$$\int_a^b w(x)\, C_n^{(\alpha)}(u(x))\, C_m^{(\alpha)}(u(x))\, dx = h_{n\alpha}\, \delta_{nm} , \qquad (4)$$

then multiplying Eq. (2) by $\frac{w(x)}{v(x)} C_m^{(\alpha)}(u(x))$ and using Eq. (4), we obtain

$$a_n = \frac{1}{h_{n\alpha}} \int_a^b f(x) \frac{w(x)}{v(x)} C_n^{(\alpha)}(u(x))\, dx \tag{5}$$

which allows the calculation of the coefficients a_n needed in Eq. (3). It is clear that the function $f(x)$ can be written as the sum of an experimentally significant part, $f_e(x)$, and a part due to optical and instrumental distortions, $f_d(x)$; that is,

$$f(x) = f_e(x) + f_d(x). \tag{6}$$

In many cases it may be convenient to further split the experimental part into an easily soluble approximate form, $f_a(x)$, and a relatively small perturbation to this form, $f_p(x)$, so that

$$f(x) = f_a(x) + [f_p(x) + f_d(x)] = f_a(x) + f_c(x) \ . \tag{7}$$

Experimentally the terms are not, of course, separable from prior knowledge, but we may arbitrarily separate out the approximate function $f_a(x)$. The same final result is obtained by performing the Abel inversion of these terms separately and then summing. Note that the polynomial expansion used for $f_a(x)$ need not be the same as that used for the two other terms of Eq. (7).

The problem then becomes one of reducing the effect of the distortion contribution, $f_d(x)$, in treating the combined contribution, $f(x)$ of Eq. (6), or $f_c(x)$ of Eq. (7). When choosing a specific form of Eq. (2) we wish therefore to satisfy two requirements:

(A) Proper weighting factor. The weighting factor in Eq. (5) should be such that the contribution to the calculation of a_n is reduced where the distortion contribution, $f_d(x)$ is largest. One would therefore like the weighting function $w(x)/v(x)$ to approach zero as $x \rightarrow 1$ and be a maximum at the center.

(B) Ease of calculation. In principle, Eq. (3) can always be evaluated numerically, but the number of integrals that must be calculated can be very large. To illustrate this point, if we wish to calculate $g(r)$ for S different values of r, then the number of integrals becomes $N(S+1)$ where N is the number of terms in the polynomial expansion. For convenience, then, Eq. (3) should be directly integrable in some closed form, or, this failing, it should be easily calculable as, for example, by a recurrence relation between the integrals of order $n+2$, $n+1$, and n. In this paper we have settled for the latter condition in order to satisfy requirement A in full.

Once the form of Eq. (2) has been set, the weighting factor of requirement A and the integrability or non-integrability of Eq. (3) in closed form are fully determined. Thus the simultaneous satisfaction of requirements A and B must be somewhat fortuitous. The number of solutions of the Abel integral equation in terms of well-known polynomials is severely limited. In rows 1 thru 4 of Table 1, we show the weighting factors and solutions of the integrals of Eq. (3) for several choices of the form of Eq. (2) where the function $v(x)$ has been chosen to allow evaluation of these integrals in closed form. All of these solutions can be derived by proper manipulation of the general equation

Table I. Solutions of the Abel Transform for some specific forms of the polynomial expansion.

Name	$v(x)C_n^{(\alpha)}(u(x))$	$w(x)/v(x)$	$1/h_{n\alpha}$	a, b	$\frac{1}{r}\frac{d}{dr}\int_r^1 \frac{v(x)\, C_n^{(\alpha)}(u(x))x\,dx}{\sqrt{x^2 - r^2}}$
Tchebycheff of 1st kind ($\alpha=0$)	$T_{2n+1}(x)/x\sqrt{1-x^2}$	x	$2/\pi$	-1, +1	$+2\pi C_{n-1}^{(3/2)}(2r^2-1)$
Legendre ($\alpha=\frac{1}{2}$)	$P_n(2x^2-1)$	x	4n+2	0, 1	$-T_{2n+1}(r)/r\sqrt{1-r^2}$
Tchebycheff of 2nd kind ($\alpha=1$)	$\sqrt{1-x^2}\, U_{2n}(x)$	1	$2/\pi$	-1, +1	$-\pi(n+\frac{1}{2})\, P_n(2r^2-1)$
Gegenbauer ($\alpha = \frac{3}{2}$)	$C_n^{(3/2)}(2x^2-1)$	$x^3(1-x^2)$	$\frac{8(2n+3)}{(n+2)(n+1)}$	0, 1	$\frac{[1+2n\, T_{2n+1}(r)/r - U_{2n}(r)]}{4(1-r^2)^{3/2}}$
Gegenbauer ($\alpha = 2$)	$\sqrt{1-x^2}\, C_{2n}^{(2)}(x)$	$(1-x^2)$	$\frac{8}{\pi(2n+3)(2n+1)}$	-1, +1	No simple closed form expression *

* Cannot be simply expressed in terms of well-known polynomials.

$$\frac{d}{r\,dr}\int_r^1 \frac{(1-x^2)^{\alpha-\frac{1}{2}}\, C_n^{(\alpha)}(2x^2-1)\mathbf{x}\,dx}{\sqrt{x^2-r^2}} =$$

$$-\sqrt{\pi}\frac{\alpha\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}(1-r^2)^{\alpha-1}\left\{C_n^{(\alpha+\frac{1}{2})}(2r^2-1)-C_{n-1}^{(\alpha+\frac{1}{2})}(2r^2-1)\right\} \quad (8)$$

We are not aware of any closed form solutions which are not specific cases derivable from this equation. The form of v(x) in these solutions is closely related to the normalization function w(x) and therefore the weighting function w(x)/v(x) cannot be arbitrarily chosen. None of the solutions listed has a weighting function of the form we desire. In this paper we present a solution of the Abel integral equation which involves a weighting function of the desired type and consists of an expansion in terms of Gegenbauer (α=2) polynomials (Table 1, Row 5).

THE SOLUTION USING GEGENBAUER (α=2) POLYNOMIALS

If we choose an expansion of the form

$$f(x) = \Sigma a_n \sqrt{1 - x^2}\, C_{2n}^{(2)}(x) \quad (9)$$

then Eq. (5) becomes

$$a_n = \frac{8}{\pi\,(2n+3)\,(2n+1)} \int_{-1}^{+1} f(x)\,(1 - x^2)\, C_{2n}^{(2)}(x)\,dx, \quad (10)$$

and we immediately see that criterion A is satisfied. For the sake of completeness, the first few polynomials of interest are given below.

$$C_0^{(2)}(x) = 1 \qquad C_2^{(2)}(x) = 12\,x^2-2 \qquad C_4^{(2)}(x) = 80\,x^4-48\,x^2+3$$

$$C_6^{(2)}(x) = 448\,x^6-480\,x^4+120\,x^2-4$$

It remains to evaluate the equation

$$g(r) = -\frac{2}{\pi}\sum_{n=0}^{N} a_n \frac{1}{r}\frac{d}{dr}\int_r^1 \frac{\sqrt{1-x^2}\,C_{2n}^{(2)}(x)\,x\,dx}{\sqrt{x^2-r^2}}. \tag{11}$$

It can be shown that

$$\sqrt{1-x^2}\,C_{2n}^{(2)}(x) = \frac{1}{4\sqrt{1-x^2}}[(2n+3)U_{2n}(x)$$

$$-(2n+1)U_{2n+2}(x)] \tag{12}$$

and therefore

$$g(r) = +\sum_{n=0}^{N} a_n [(2n+3) I_n(r) - (2n+1) I_{n+1}(r)], \tag{13}$$

where

$$I_n(r) \equiv \frac{-1}{2\pi}\frac{1}{r}\frac{d}{dr}\int_r^1 \frac{U_{2n}(x)\,x\,dx}{\sqrt{1-x^2}\sqrt{x^2-r^2}} \tag{14}$$

and thus $I_o(r) = 0$ and $I_1(r) = -1.$ (15)

Substituting the recurrence relation

$$U_{2n+4}(x) = 2(2x^2-1)U_{2n+2}(x) - U_{2n}(x)$$

$$= +2U_{2n+2}(x) - U_{2n}(x) + 4(x^2-1)U_{2n+2}(x) \tag{16}$$

into Eq.(14),we obtain

$$I_{n+2} = 2I_{n+1} - I_n + \frac{2}{\pi}\frac{1}{r}\frac{d}{dr}\int_r^1 \frac{\sqrt{1-x^2}\,U_{2n+2}(x)\,x\,dx}{\sqrt{x^2-r^2}}. \tag{17}$$

The integral of the last term has a closed form solution (see Table 1) so that Eq. (17) becomes

$$I_{n+2}(r) = 2\, I_{n+1}(r) - I_n(r) - (2n+3)P_{n+1}(2r^2 - 1), \qquad (18)$$

where $P_{n+1}(2r^2 - 1)$ is also generated by a recurrence relation[5] given by

$$P_{n+1}(2r^2 - 1) = \frac{2n+1}{n+1}(2r^2 - 1)\, P_n(2r^2 - 1)$$

$$- \frac{n}{n+1} P_{n-1}(2r^2 - 1). \qquad (19)$$

Using Eqs. (15), (18), and (19), the desired function, g(r), can readily be obtained from Eq. (13).

This approach was readily programmed for a computer solution. The integral of Eq. (10) was evaluated using the trapezoidal rule and a 40 point Gaussian quadrature. The change in the values of the coefficients using the trapezoidal rule was completely negligible. A listing of the Fortran computer program using the trapezoidal rule is given in Appendix II. The series of coefficients so calculated can be terminated by applying the F-distribution significance test[6] to each coefficient in turn. We start by assuming that a_o is always significant. We then calculate the mean square deviation, δ, between the input curve $f(x_i)$ and the curve defined by Eq. (2) using the coefficients known to be significant. This calculation, when repeated using one additional coefficient to be tested, yields δ'. The quantity $F^* = (\delta - \delta') \times$ (number of points in the input curve)/δ' is then compared with the value of F chosen from

Table V of Ref. 6. If $F^* > F$ then the additional coefficient is considered significant and the procedure is repeated for the following coefficient. When a non-significant coefficient is found, all subsequent coefficients are then set equal to zero. This is done on the physical basis that fine structure is not expected.

In treating the input curves we allowed for asymmetries by following the approach of Freeman and Katz,[1] and thus did not force the transformed curves to be symmetric. This was done, in spite of the fact that no large asymmetries were expected, in order to obtain a check on the symmetry of the discharge. In this approach, the symmetric part of f(x) is contained in the even function

$$f_g(x) = \frac{f(+x) + f(-x)}{2} \tag{20}$$

and the asymmetric part of f(x) is described by the even function

$$f_u(x) = \frac{f(+x) - f(-x)}{2x} \quad . \tag{21}$$

After the Abel inversion of these functions to obtain $g_g(r)$ and $g_u(r)$, the radial distribution is constructed by using

$$g(+r) = g_g(r) + rg_u(r) \tag{22}$$

and

$$g(-r) = g_g(r) - rg_u(r). \tag{23}$$

CONCLUSION

The use and advantages of orthogonal polynomial expansions in solving the Abel integral transform have been briefly discussed and a new solution in terms of Gegenbauer ($\alpha = 2$) polynomials, which utilizes a weighting function $(1-x^2)$ in the calculation of the expansion coefficients, has been presented. In the presence of significant distortions near the boundary of the luminous region, a condition which is particularly true for projected profiles approximating $f(x) = \quad 1-x^2$, or $g(r)$ approximately constant, the use of the Gegenbauer ($\alpha=2$) polynomial expansion will reduce the contribution of such distortions in the calculation of the expansion coefficients, and in principle allow a more accurate solution of the Abel inversion integral.

ACKNOWLEDGEMENT

We wish to thank Dr. John B. Shumaker for supplying us with references and information on his computer program for performing Abel inversions.

REFERENCES

1. M. P. Freeman and S. Katz, J. Opt. Soc. Am. 50, 826 (1960); 53, 1172 (1963).

2. S. I. Herlitz, Technical Note #5 Institute of Physics, Univ. of Uppsala, Uppsala, Sweden (July 31, 1961); Arkiv Fysik 23, 571 (1963).

3. G. H. Popenoe and J. B. Shumaker, NBS J. Res. 69A, 495 (1965).

4. For an axially symmetric light source there is no distortion due to refraction in passing thru a glass wall of uniform thickness. The azimuthal angle of a light ray is changed but the "impact parameters" inside and outside of the tube are equal, to the extent that the index of refraction inside is equal to that of air (see Appendix I).

5. M. Abramowitz and Irene A. Stegun, Handbook of Mathematical Functions, NBS Applied Mathematics Series 55 (June 1964), in particular p. 782.

6. Paul G. Hoel, Introduction to Mathematical Statistics (Wiley, 1954).

APPENDIX I

We present here a proof of the statement that the "impact parameter" of a light ray is unchanged by passage thru the wall of a uniform glass tube. Referring to figure 1, we write the law of sines for triangle ABC as $\sin\theta_2 = \sin(180-\theta_3)/(1+\tau)$, or

$$(1+\tau)\sin\theta_2 = \sin\theta_3. \quad \text{(A-1)}$$

The initial and final impact parameters are given by the relations

$$p_i = (1+\tau)\sin\theta_1 \text{, and } p_f = \sin\theta_4. \quad \text{(A-2)}$$

The angles involved are related by Snell's law as follows:

$$\sin\theta_1/\sin\theta_2 = \sin\theta_4/\sin\theta_3 = \eta. \quad \text{(A-3)}$$

Here η is the index of refraction of the glass tube relative to that of air and the indices of refraction both inside and outside of the tube are assumed identical. Combining these relations, we obtain $p_f = \sin\theta_4 = \sin\theta_1 (\sin\theta_3/\sin\theta_2) = \sin\theta_1 (1+\tau) = p_i$. Thus the impact parameter inside the tube is identical with that of the incident ray. This result is simply another expression of the conservation of angular momentum. There is consequently no distortion due to refraction for the case of a cylindrically symmetric light source, coaxial with a cylindrical glass tube of uniform thickness.

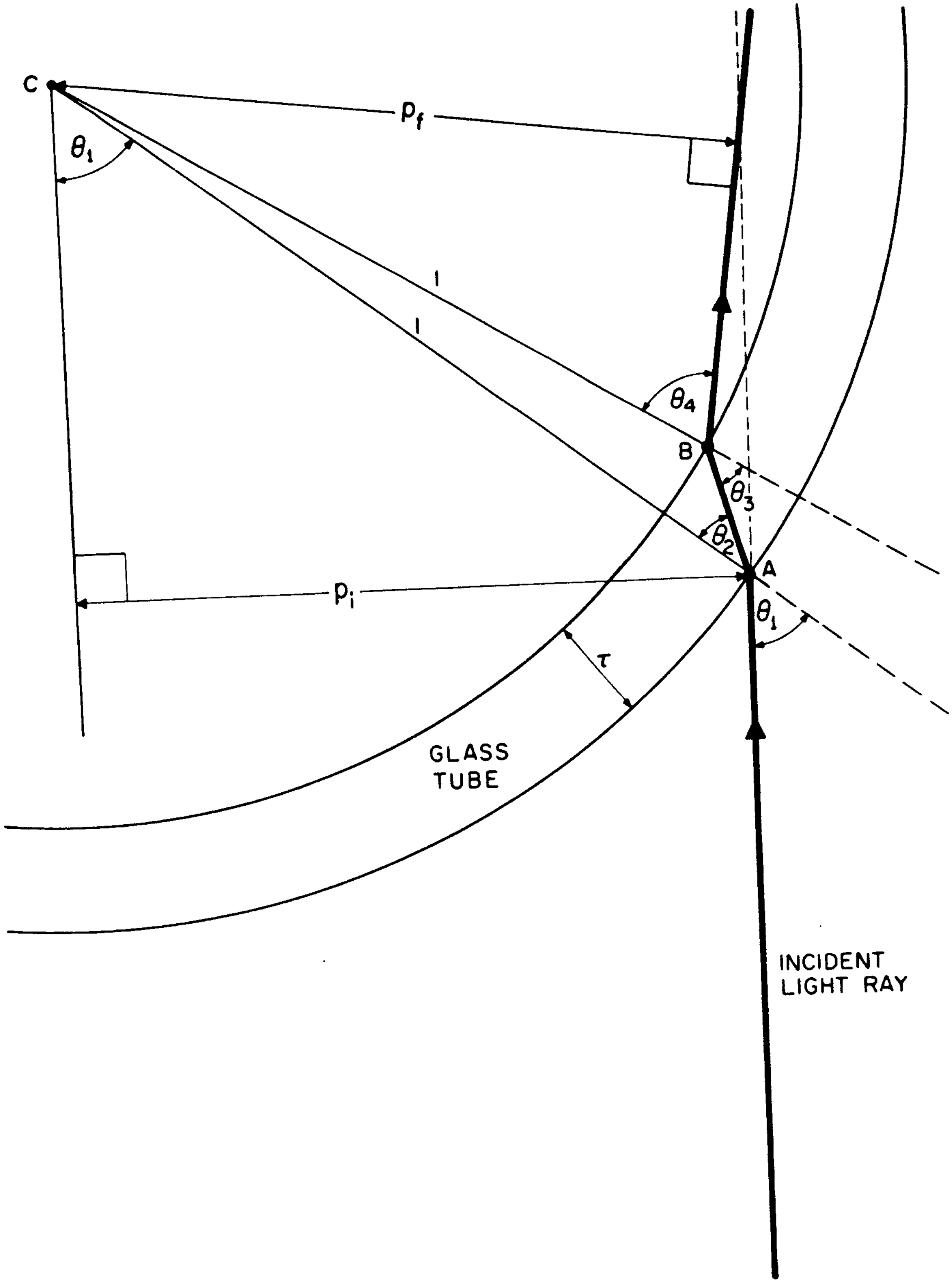

Figure 1. Displacement of an incident beam of light in passing through an optically perfect glass tube. The "impact parameter" is not changed.

APPENDIX II

```
      PROGRAM ABELTAPE
      DIMENSION HED( 5),X(500),Y(500),RP(100),RM(100),XP(250),YP(250),
     1YM(250),XF(501),FX(501),UR1(501),UR2(501),YKN(251),AN(20),BN(20),
     2ANN(20),BNN(20),RA(20),RB(20),RX(501),CS(501),BS(501),GP(100),
     3GM(100),KIND(3,2),GR(201),XJP(251),XJM(251),XX(251),HEE(10)
      TYPE INTEGER DATE
      COMMON NPOINT
      DATA(IRUNO=0),(ISETO=0)
      DATA(ESPI=5.092958),(FSP=1.2732395),((KIND(1,I),I=1,2)=8H   TCHEB,
     15HICHEF),((KIND(2,I),I=1,2)=8H    LEGE,4HNDRE),((KIND(3,I),I=1,2)=
     28H    GEGE,5HNBAUR)
  112 FORMAT(*0SET=*,I2,5X,*RUN=*,I2,*  NOT FOUND ON INPUT TAPE*)
      READ(60,505)HEE
  505 FORMAT(10A8)
C
C               GENERATE RHO ARRAY
C
      RZ=0.
      RP(1)=.01
      RM(1)=-.01
      DO 106 I=2,100
      RP(I)=RP(I-1)+.01
  106 RM(I)=-RP(I)
      DATE=IDATE(XDUMMY)
C
C               GENERATE GR ARRAY
C
      DO 500 I=1,100
      IM=101-I
      IP=101+I
      GR(I)=RM(IM)
  500 GR(IP)=RP(I)
      GR(101)=0.
C
C               READ INPUT PARAMETERS,DATA, AND HEADING.
C
   99 READ(60,111)ISET,IRUN,NN,NX,HED,INFLAG
  111 FORMAT(4I5,5A8,I1)
      IF(EOF,60)98,501
  501 IF(INFLAG.EQ.0)  GO TO 503
      READ(60,153)(X(I),Y(I),I=1,NX)
  153 FORMAT(8(F5.0,F5.1))
      GO TO 9999
  503 READ(60,151)(X(I),I=1,NX),(Y(I),I=1,NX)
  151 FORMAT(8E10.3)
 9999 READ(60,152)NPOINT,NP,FVALUE,THICK,IPPP
  152 FORMAT(2I5,2F10.5,I1)
C
C               GENERATE ARRAYS FOR YM AND YP, THE CORRESPONDING Y
C               VALUES FOR -X AND +X RESPECTIVELY AND ASSUME THE
C               VALUES AT THE END POINTS ARE AVERAGES OF THE MEASURED Y.
C               NH=NX/2 FOR EVEN NX.
C               NH=(NX-1)/2 FOR ODD NX.
C               NT IS THE TOTAL NO. OF POINTS.
```

```
C
C              FIRST FIND THE AVERAGE DELTA X.
C
    6 DXA=2./FLOATF(NX-1)
C
C              FIND N= NX  MODULO  2.
C
      N=XMODF(NX,2)
      IF(N.EQ.0)12,13
C
C              N=0 IMPLIES NX IS EVFN.
C
   12 NH=NX/2
      DXH=DXA/2.
      NHPO=NH+1
      DO 14 I=1,NH
      NHP=NH+I
      YP(I)=Y(NHP)
      XP(I)=DXA*FLOATF(I-1)+DXH
      NHM=NHPO-I
   14 YM(I)=Y(NHM)
      YZ=.5*(Y(NH)+Y(NHPO))
      GO TO 16
C
C              N NOT 0 IMPLIES NX IS ODD.
C
   13 NH=(NX-1)/2
      NHPO=NH+1
      DO 15 I=1,NH
      NHP=NHPO+I
      YP(I)=Y(NHP)
      NHM=NHPO-I
      XP(I)=DXA*FLOATF(I)
   15 YM(I)=Y(NHM)
      YZ=Y(NHPO)
      DXH=DXA
   16 XP(NH)=1.
      YA=.5*(YP(NH)+YM(NH))
      YP(NH)=YA
      YM(NH)=YA
C
C              APPLY BACKGROUND NOISE CORRECTION TO THE DATA.
C
      TP=THICK+1.
      TPS=TP*TP
      IF(THICK.EQ.0.)  C=0.
      IF(THICK.NE.0.)  C=YA/SQRT(TPS-1.)
      YZ=YZ-C*THICK
      DO 17 I=1,NH
      XSQ=XP(I)*XP(I)
      DXP=SQRTF(TPS-XSQ)-SQRTF(1.-XSQ)
      CD=C*DXP
      YP(I)=YP(I)-CD
   17 YM(I)=YM(I)-CD
```

```
C
C           TRANSFORM THE FUNCTION SO THAT IT IS ZERO AT THE
C           BOUNDARY AND IT IS NORMALIZED.
C
      YZ=YZ-YA
      YMAXIN=YZ
      DO 18 I=1,NH
      YP(I)=YP(I)-YA
      YM(I)=YM(I)-YA
      IF(YP(I).GT.YMAXIN)19,21
   19 YMAXIN=YP(I)
   21 IF(YM(I).GT.YMAXIN)22,18
   22 YMAXIN=YM(I)
   18 CONTINUE
      YZ=YZ/YMAXIN
      DO 20 I=1,NH
      YP(I)=YP(I)/YMAXIN
   20 YM(I)=YM(I)/YMAXIN
C
C           SET UP XF AND FX ARRAYS USED BY THE
C           SIGNIFICANT COEFFICIENT TEST.
C
      NHPO=NH+1
      NT=NH+NHPO
      DO 23 I=1,NH
      NHM=NHPO-I
      FX(I)=YM(NHM)
      XF(I)=-XP(NHM)
      NHP=NHPO+I
      FX(NHP)=YP(I)
   23 XF(NHP)=XP(I)
      FX(NHPO)=YZ
      XF(NHPO)=0.
C
C           CALCULATE SYMMETRIC AND ASYMMETRIC PORTIONS OF FUNCTION.
C
      NHPO=NH+1
      XJP(1)=YZ
      XJM(1)=0.
      DO 24 I=2,NHPO
      XJP(I)=.5*(YP(I-1)+YM(I-1))
   24 XJM(I)=(YP(I-1)-YM(I-1))/(2.*XP(I-1))
C
C           WRITE OUT FIRST PAGE OF OUTPUT.
C
      WRITE(61,121)HEE,HED,DATE,ISET,IRUN,NN,NX,FVALUE,THICK,NP,KIND(NPO
     1INT,1),KIND(NPOINT,2)
  121 FORMAT(1H1,10A8,5A8,3X,A8                //6H   SET=I3,4X,4HRUN=I3,4X
     1,*TOTAL NO POINTS=*,I4,4X,*NO POINTS USED=*,I4,4X,8HF VALUE=F6.2,4
     2X,6HTHICK=F8.3,4X,3HFORI3,1X,2A8,6H TERMS//)
      WRITE(61,122)
  122 FORMAT(* THE LISTING OF THE INPUT DATA FOLLOWS*//)
      WRITE(61,123)
  123 FORMAT(5(10X,1HX,8X,1HY,2X))
```

```
      WRITE(61,124)(X(I),Y(I),I=1,NN)
  124 FORMAT(5(F13.4,F9.4))
      WRITE(61,121)HEE,HFD,DATE,ISFT,IRUN,NN,NX,FVALUE,THICK,NP,KIND(NPO
     1INT,1),KIND(NPOINT,2)
      WRITE(61,125)
  125 FORMAT(* THE LISTING OF THE NORMALIZED INPUT DATA FOLLOWS*//)
      WRITE(61,123)
      WRITE(61,124)(XF(I),FX(I),I=1,NT)
C     GENERATION OF THE COEFFICIENTS FOR THE DESIRED POLYNOMIAL
      DXH=DXH/2.
      XX(1)=0.
      DO 91 I=2,NHPO
   91 XX(I)=XP(I-1)
      GO TO (100,200,300),NPOINT
C                                     NPOINT=1, TCHEBICHEF CASE
  100 DO 101 I=1,NHPO
      UR2(I)=1.
      UR1(I)=XX(I)+XX(I)
      IF(I.EQ.1)198,199
  198 AN(1)=XJP(1)*DXH
      BN(1)=XJM(1)*DXH
      GO TO 101
  199 AN(1)=AN(1)+XJP(I)*DXA
      BN(1)=BN(1)+XJM(I)*DXA
  101 CONTINUE
      AN(1)=AN(1)*FSP
      BN(1)=BN(1)*FSP
      DO 102 J=2,NP
      DO 103 I=1,NHPO
      TX=XX(I)+XX(I)
      UR2(I)=TX*UR1(I)-UR2(I)
      UR1(I)=TX*UR2(I)-UR1(I)
      IF(I.EQ.1)104,105
  104 URR=UR2(I)*DXH
      AN(J)=XJP(I)*URR
      BN(J)=XJM(I)*URR
      GO TO 103
  105 URR=UR2(I)*DXA
      AN(J)=AN(J)+XJP(I)*URR
      BN(J)=BN(J)+XJM(I)*URR
  103 CONTINUE
      AN(J)=AN(J)*FSP
      BN(J)=BN(J)*FSP
  102 CONTINUE
      GO TO 11
C                                     NPOINT=2, LEGENDRE CASE
  200 DO 201 I=1,NHPO
      UR2(I)=1.
      YKN(I)=2.*XX(I)*XX(I)-1.
      UR1(I)=YKN(I)
      IF(I.EQ.1)298,299
  298 URR=DXH*XX(1)
      AN(1)=XJP(1)*URR
      BN(1)=XJM(1)*URR
```

```
      GO TO 201
  299 URR=DXA*XX(I)
      AN(1)=AN(1)+XJP(I)*URR
      BN(1)=BN(1)+XJM(I)*URR
  201 CONTINUE
      AN(1)=2.*AN(1)
      BN(1)=2.*BN(1)
      DO 202 J=2,NP
      IF(J.EQ.2)234,235
  234 XN=J-1
      GO TO 231
  235 XN=J-2
      XN1=J-1
  231 XN2=XN+XN+1.
      DO 203 I=1,NHPO
      IF(J.EQ.2)244,245
  244 UR=UR1(I)
      GO TO 249
  245 UR=(XN2*YKN(I)*UR1(I)-XN*UR2(I))/XN1
      UR2(I)=UR1(I)
      UR1(I)=UR
  249 IF(I.EQ.1)204,205
  204 URR=UR*DXH*XX(I)
      AN(J)=XJP(I)*URR
      BN(J)=XJM(I)*URR
      GO TO 203
  205 URR=UR*DXA*XX(I)
      AN(J)=AN(J)+XJP(I)*URR
      BN(J)=BN(J)+XJM(I)*URR
  203 CONTINUE
      IF(J.EQ.2)254,255
  255 XN2=XN2+2.
  254 TX=XN2*2.
      AN(J)=AN(J)*TX
      BN(J)=BN(J)*TX
  202 CONTINUE
      GO TO 11
C                                           NPOINT=3, GEGENBAUR CASE
  300 DO 301 I=1,NHPO
      YKN(I)=1.-XX(I)*XX(I)
      UR2(I)=1.
      UR1(I)=4.*XX(I)
      IF(I.EQ.1)398,399
  398 URR=YKN(1)*DXH
      AN(1)=XJP(1)*URR
      BN(1)=XJM(1)*URR
      GO TO 301
  399 URR=YKN(I)*DXA
      AN(1)=AN(1)+XJP(I)*URR
      BN(1)=BN(1)+XJM(I)*URR
  301 CONTINUE
      TX=ESPI/3.
      AN(1)=AN(1)*TX
      BN(1)=BN(1)*TX
```

```
      XN=0.
      DO 302 J=2,NP
      XN=XN+1.
      XN1=XN+1.
      XN2=XN1+1.
      XN3=XN2+1.
      XN4=XN3+1.
      TXRAT=XN2*XN4
      DO 303 I=1,NHPO
      UR2(I)=(2.*XN2* XX(I)*UR1(I)-XN3*UR2(I))/XN1
      UR1(I)=(2.*XN3* XX(I)*UR2(I)-XN4*UR1(I))/XN2
      UR=UR2(I)*YKN(I)
      IF(I.EQ.1)304,305
  304 URR=UR*DXH
      AN(J)=XJP(I)*URR
      BN(J)=XJM(I)*URR
      GO TO 303
  305 URR=UR*DXA
      AN(J)=AN(J)+XJP(I)*URR
      BN(J)=BN(J)+XJM(I)*URR
  303 CONTINUE
      TX=ESPI/TXRAT
      AN(J)=AN(J)*TX
      BN(J)=BN(J)*TX
  302 CONTINUE
C
C                 TEST FOR SIGNIFICIENT COEFFICIENTS.
C
   11 DO 31 I=1,NP
      ANN(I)=AN(I)
      BNN(I)=BN(I)
      RA(I)=0.
   31 RB(I)=0.
      RA(1)=FVALUE+1.
      GO TO (32,33,32),NPOINT
   33 DO 34 K=1,NT
   34 RX(K)=1.
      GO TO 35
   32 DO 36 K=1,NT
   36 RX(K)=SQRTF(1.-XF(K)*XF(K))
   35 DO 37 I=1,NP
      IF(I.EQ.1)38,39
   38 DO 41 J=1,NT
      UR1(J)=1.
      UR2(J)=1.
   41 BS(J)=AN(1)
      GO TO 26
   39 IF(I.EQ.2)43,44
   43 DO 45 J=1,NT
      GO TO (51,52,53),NPOINT
   51 UR2(J)=4.*XF(J)*XF(J)-1.
      UR1(J)=XF(J)+XF(J)
      GO TO 54
   52 Z=2.*XF(J)*XF(J)-1.
```

```
      UR2(J)=Z
      GO TO 54
   53 UR2(J)=12*XF(J)*XF(J)-2.
      UR1(J)=4.*XF(J)
   54 CS(J)=BS(J)
   45 BS(J)=BS(J)+AN(I)*UR2(J)*RX(J)
      GO TO 46
   44 DO 47 J=1,NT
      GO TO (61,62,63),NPOINT
   61 TXX=XF(J)+XF(J)
      UR1(J)=TXX*UR2(J)-UR1(J)
      UR2(J)=TXX*UR1(J)-UR2(J)
      GO TO 64
   62 XM2=I
      XMN=I-1
      XM1=XM2+XMN
      Z=2.*XF(J)*XF(J)-1.
      TXX=(Z*UR2(J)*XM1-XMN*UR1(J))/XM2
      UR1(J)=UR2(J)
      UR2(J)=TXX
      GO TO 64
   63 XM2=I+I-3
      XM3=XM2+1.
      XM4=XM3+1.
      XM5=XM4+1.
      UR1(J)=(2.*XM3*XF(J)*UR2(J)-XM4*UR1(J))/XM2
      UR2(J)=(2.*XM4*XF(J)*UR1(J)-XM5*UR2(J))/XM3
   64 CS(J)=BS(J)
   47 BS(J)=BS(J)+AN(I)*UR2(J)*RX(J)
   46 AS=0.
      DS=0.
      DO 48 K=1,NT
      ASR=FX(K)-CS(K)
      DSR=FX(K)-BS(K)
      AS=AS+ASR*ASR
   48 DS=DS+DSR*DSR
      RA(I)=ABSF(100.*(AS-DS)/DS)
   26 AS=0.
      DS=0.
      DO 68 K=1,NT
      CS(K)=BS(K)
      BS(K)=BS(K)+BN(I)*UR2(K)*XF(K)*RX(K)
      ASR=FX(K)-CS(K)
      DSR=FX(K)-BS(K)
      AS=AS+ASR*ASR
   68 DS=DS+DSR*DSR
      RB(I)=ABSF(100.*(AS-DS)/DS)
   37 CONTINUE
      IPP=1
      DO 42 I=1,NP
      IF(RA(I).LE.FVALUE)65,69
   65 DO 66 J=I,NP
      ANN(J)=0.
   66 BNN(J)=0.
```

```
      GO TO 85
   69 GO TO (84,42),IPP
   84 IF(RB(I).LE.FVALUE)71,42
   71 DO 72 J=I,NP
   72 BNN(J)=0.
      IPP=2
   42 CONTINUE
   85 IPP=1
C
C                 GENERATE F(RHO) VERSUS RHO.
C
   67 CALL FGRHO(RZ,ANN,1,GZ,NP)
      DO 73 I=1,100
      CALL FGRHO(RP(I),ANN,1,GP(I),NP)
      CALL FGRHO(RM(I),BNN,2,GM(I),NP)
   73 CONTINUE
C
C                 WRITE OUT SECOND PAGE OF OUTPUT.
C
      WRITE(61,121)HEE,HED,DATE,ISET,IRUN,NN,NX,FVALUE,THICK,NP,KIND(NPO
     1INT,1),KIND(NPOINT,2)
      WRITE(61,126)
  126 FORMAT(8X,*SIG COEFF*,21X,*ALL COEFF*,24X,*RATIOS*/4X,*A(I)*,9X,*B
     1(I)*,13X,*A(I)*,9X,*B(I)*,13X,*RA(I)*,10X,*RB(I)*//)
      WRITE(61,127)(ANN(I),BNN(I),AN(I),BN(I),RA(I),RB(I),I,I=1,NP)
  127 FORMAT(F10.6,F13.6,F17.6,F13.6,E19.4,E15.4,6X,I2)
C
C                 NORMALIZE F(RHO) AND SET TO ZERO IF IT IS NEGATIVE.
C
   81 DO 74 I=1,100
      IM=101-I
      IP=101+I
      FX(I)=GP(IM)-GM(IM)
   74 FX(IP)=GP(I)+GM(I)
      FX(101)=GZ
      FMXOUT=FX(1)
      DO 75 I=2,201
      IF(FX(I).GT.FMXOUT)76,75
   76 FMXOUT=FX(I)
   75 CONTINUE
   83 DO 77 I=1,201
      FX(I)=FX(I)/FMXOUT
      IF(FX(I).LT.0.)78,77
   78 FX(I)=0.
   77 CONTINUE
C
C                 WRITE OUT THIRD PAGE OF OUTPUT.
C
      WRITE(61,121)HEE,HED,DATE,ISET,IRUN,NN,NX,FVALUE,THICK,NP,KIND(NPO
     1INT,1),KIND(NPOINT,2)
      WRITE(61,128)
  128 FORMAT(5(9X,3HRHO,4X,6HF(RHO))//)
      WRITE(61,124)(GR(I),FX(I),I=1,201)
      WRITE(61,129)YMAXIN,FMXOUT
```

```
129 FORMAT(*0THE NORMALIZATION FACTORS ARE*//* INPUT/*,E11.4,10X,*OUTP
   1UT/*,E11.4)
    IF(IPPP.EQ.1)9999,99
 98 WRITE(61,131)
131 FORMAT(* END OF THIS RUN*)
    WRITE(61,506)HEE,DATE
506 FORMAT(1H1,10A8,20X,A8)
    CALL EXIT
    END
    SUBROUTINE FGRHO(X,AN,J,FOX,NP)
    DIMENSION AN(20)
    COMMON NPOINT
    DATA(TP=6.2831853)
    Y=2.*X*X-1.
    XX=3.
    GO TO (11,12,13),NPOINT
 11 P2=1.
    P1=Y
    S=AN(  1)+3.*AN(  2)*Y
    XN=0.
    DO 1 I=3,NP
    XX=XX+2.
    XN=XN+1.
    PN=((2.*XN+1.)*Y*P1-XN*P2)/(XN+1.)
    S=S+XX*AN(  I)*PN
    P2=P1
    P1=PN
  1 CONTINUE
    GO TO (2,3),J
  2 FOX=.5*S
    RETURN
  3 FOX=-.5*S*X
    RETURN
 12 XSQ=X*X
    ZSQ=1.-XSQ
    TZSQ=ZSQ+ZSQ
    Z=SQRTF(ZSQ)
    P2=1.
    P1=3.-4.*XSQ
    S=AN(  1)-AN(  2)*P1
    SI=1.
    DO 4 I=3,NP
    PN=TZSQ*P1-P2
    P2=P1
    P1=PN
    S=S+SI*AN(  I)*PN
  4 SI=-SI
    DEN=1.5707963*Z
    GO TO (5,6),J
  5 FOX=S/DEN
    RETURN
  6 FOX=-S*X/DEN
    RETURN
 13 P2=1.
```

```
      P1=Y
      T2=-1.
      T1=1.-6.*X*X
      S=AN(   1)+AN(   2)*(5.*T2-3.*T1)
      DO 8 I=3,NP
      XN=I-1
      XNN=XN-1.
      XB=XN+XN+1.
      XA=XB+2.
      PN=((XN+XNN)*Y*P1-XNN*P2)/XN
      TN=T1+T1-XB*PN-T2
      S=S+AN(   I)*(XA*T1-XB*TN)
      T2=T1
      T1=TN
      P2=P1
    8 P1=PN
      GO TO (7,9),J
    9 FOX=    S*X
      RETURN
    7 FOX=S
      RETURN
      END
```


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