

# Simulations of the Hadamard Variance: Probability Distributions and Confidence Intervals

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**Abstract**—Power-law noise in clocks and oscillators can be simulated by Fourier transforming a modified spectrum of white phase noise. This approach has been applied successfully to simulation of the Allan variance and the modified Allan variance in both overlapping and nonoverlapping forms. When significant frequency drift is present in an oscillator, at large sampling times the Allan variance overestimates the intrinsic noise, while the Hadamard variance is insensitive to frequency drift. The simulation method is extended in this paper to predict the Hadamard variance for the common types of power-law noise. Symmetric real matrices are introduced whose traces—the sums of their eigenvalues—are equal to the Hadamard variances, in overlapping or nonoverlapping forms, as well as for the corresponding forms of the modified Hadamard variance. We show that the standard relations between spectral densities and Hadamard variance are obtained with this method. The matrix eigenvalues determine probability distributions for observing a variance at an arbitrary value of the sampling interval  $\tau$ , and hence for estimating confidence in the measurements.

**Index Terms**—Analysis of variance, digital simulation, time series analysis.

## I. INTRODUCTION

WHEN an oscillator's frequency drifts, usually the long-term behavior of the Allan variance is dominated by the drift, and the oscillator stability is not well characterized by the Allan variance [3]. We could approach this problem by estimating and removing the drift from the measured time series, but the estimation process may itself introduce uncertainties. If the time series is only of the order of a day or so in length, drift removal may not introduce intolerable uncertainties [4]. However, removing the drift from longer time series may also remove very long-term random components that might result in severe underestimation of the variance for large sampling times. The Hadamard variance, which is defined (see below) in terms of a third difference of values of the time measurements, is naturally insensitive to drift and is commonly applied to clocks such as those based on rubidium atomic frequency standards (RAFS), which are known to suffer from unpredictable frequency drift following launch but which have better stability than cesium clocks [6], [24]. In this paper, we build on our previous work that delineates the simulation of power-law noises for the Allan variance and the modified Allan variance [1], [2].

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Imperfections in performance of the clock under test are studied by analyzing noise in the time deviation sequence  $x_k = t_k - k\tau_0$ , or the fractional frequency difference during the sampling interval  $\tau = s\tau_0$

$$\Delta_{k,s}^{(1)} = (x_{k+s} - x_k)/(s\tau_0). \quad (1)$$

The frequency spectrum of fractional frequency differences can usually be adequately characterized by linear superposition of a small set of types of power-law noise. The frequency spectrum of the fractional frequency differences of a particular noise type is given by a one-sided spectral density [5]

$$S_y(f) = h_\alpha f^\alpha, \quad f > 0 \quad (2)$$

where the units of  $S_y(f)$  are  $\text{Hz}^{-1}$ . For the common power-law noise types,  $\alpha$  varies in integral steps from +2 down to -2 corresponding, respectively, to white phase modulation, flicker phase modulation, white frequency modulation, flicker frequency modulation, and random walk of frequency.

This paper is organized as follows. Section II summarizes the basic simulation method, and Section III applies the method to the overlapping Hadamard variance. Analytic expressions for limiting values of overlapping Hadamard variances at large sampling times are given in Section IV. The nonoverlapping Hadamard variance is described in Section V, and the modified overlapping case is presented in Section VI. Section VII shows how diagonalization of the averaged squared second-difference operator, applied to the simulated time series, leads to expressions for the probability of observing a value of the variance for some chosen value of the sampling or averaging time. The approach is extended to discuss the modified nonoverlapping Hadamard variance in Section VIII. Application to radar variance, as an example, is discussed in Section IX, followed by summary and conclusion in Section X.

## II. DISCRETE TIME SERIES

The noise amplitudes at Fourier frequencies  $f_m$  may be represented by a set of  $N$  normally distributed random complex numbers  $w_n$  having mean zero and variance  $\langle w_m^* w_m \rangle = 2\sigma^2$  that would by themselves generate a simulated spectrum for white phase noise [1]

$$X_k = \frac{\tau_0^2 \sigma}{\sqrt{N}} \sum_{m=-N/2+1}^{N/2} \frac{|f_0|^\lambda}{|f_m|^\lambda} e^{-\frac{2\pi i m k}{N}} \left( \frac{w_m}{\sigma} \right). \quad (3)$$

The factor  $\tau_0^2$  is introduced in (3) so that when the random numbers  $w_m$  are interpreted as frequencies,  $X_k$  has physical

dimensions of time. The negative sign for the exponent in (3) was chosen to simplify later equations. In (3), the discrete frequencies are

$$f_m = \frac{m}{N\tau_0}, \quad -\frac{N}{2} + 1 \leq m \leq \frac{N}{2} \quad (4)$$

with  $N$  being an even number. The frequency parameter  $f_0$  was also introduced to make the physical dimensions of (3) consistent. It is a constant unrelated to (4). The product  $\tau_0^2 f_0^\lambda$  determines the level  $h_\alpha$  of the noise. An alternative expression involving a sum over positive frequencies only is

$$X_k = \frac{2\tau_0^2}{\sqrt{N}} \sum_{m=1}^{N/2-1} \frac{|f_0|^\lambda}{|f_m|^\lambda} \left( \cos \frac{2\pi mk}{N} u_m + \sin \frac{2\pi mj}{N} v_m \right) + \frac{\tau_0^2}{\sqrt{N}} \frac{|f_0|^\lambda}{|f_{N/2}|^\lambda} (-1)^k u_{N/2}. \quad (5)$$

We shall assume that the variance of the noise amplitudes is such that

$$\langle (w_m)^* w_n \rangle = \langle u^2 + v^2 \rangle \delta_{mn} = 2\sigma^2 \delta_{mn}; \quad m \neq 0, N/2. \quad (6)$$

$w_m = u_m + iv_m$  where  $u_m$  and  $v_m$  are independent uncorrelated random numbers. Also,  $\langle u^2 - v^2 + 2iuv \rangle = 0$  for  $m \neq N/2$ . In order to avoid division by zero, we shall always assume that the Fourier amplitude corresponding to zero frequency vanishes. This only means that the average of the time residuals in the time series will be zero, and has no effect on any variance that involves time differences.

To obtain the correct spectral density, we shall assume that the constants introduced in (5) are related to the strength of the power-law noise by

$$\frac{\tau_0^2 |f_0|^\lambda}{\sqrt{N}} = \left( \frac{h_\alpha}{16\pi^2 \sigma^2 N \tau_0} \right)^{1/2}. \quad (7)$$

It has been shown [1] that if  $2\lambda = 2 - \alpha$ , the correct average spectral density is obtained (e.g., white phase noise is generated when  $\lambda = 0$ ). The simulated time series is

$$X_k = \left( \frac{h_\alpha}{16\pi^2 N \tau_0} \right)^{1/2} \sum_m \frac{e^{-\frac{2\pi i m k}{N}}}{|f_m|^{1-\alpha/2}} \left( \frac{w_m}{\sigma} \right). \quad (8)$$

### III. OVERLAPPING HADAMARD VARIANCE

Consider the third-difference operator defined by

$$\Delta_{j,s}^{(3)} = \frac{1}{\sqrt{6\tau^2}} (X_{j+3s} - 3X_{j+2s} + 3X_{j+s} - X_j). \quad (9)$$

The completely overlapping Hadamard variance is formed by averaging the square of this third difference over all possible values of  $j$  from 1 to  $N - 3s$ : Thus

$$\sigma_{Ho}^2(\tau) = \frac{1}{N - 3s} \left\langle \sum_{j=1}^{N-3s} \left( \Delta_{j,s}^{(3)} \right)^2 \right\rangle. \quad (10)$$

In terms of the time series (8), the third difference can be reduced using elementary trigonometric identities

$$\begin{aligned} \Delta_{j,s}^{(3)} &= \left( \frac{h_\alpha}{96\tau^2 \pi^2 \sigma^2 N \tau_0} \right)^{1/2} \sum_m \frac{w_m}{|f_m|^\lambda} e^{-\frac{\pi i m (2j+3s)}{N}} \\ &\quad \times \left( e^{-\frac{3\pi i m s}{N}} - 3e^{-\frac{\pi i m s}{N}} + 3e^{\frac{\pi i m s}{N}} - e^{\frac{3\pi i m s}{N}} \right) \\ &= i \left( \frac{2h_\alpha}{3\tau^2 \pi^2 \sigma^2 N \tau_0} \right)^{1/2} \sum_m \frac{w_m e^{-\frac{\pi i m (2j+3s)}{N}}}{|f_m|^\lambda} \\ &\quad \times \left( \sin \left( \frac{\pi m s}{N} \right) \right)^3. \end{aligned} \quad (11)$$

We form the averaged square of  $\Delta_{j,s}^{(3)}$  by multiplying the expression (11) by its complex conjugate, then summing over all possible values of  $j$  and averaging. After averaging, only terms corresponding to the same frequencies in the two factors contribute. the overlapping Hadamard variance is

$$\sigma_{Ho}^2(\tau) = \frac{4h_\alpha}{3N\tau_0\pi^2\tau^2} \sum_m \left( \sin \left( \frac{\pi m s}{N} \right) \right)^6 \frac{1}{|f_m|^{2\lambda}}. \quad (12)$$

The spacing between frequencies is  $1/(N\tau_0) = df$ ; in the limit of large  $N$ , the sum over frequencies passes to an integral

$$\sigma_{Ho}^2(\tau) = \frac{4h_\alpha}{3\pi^2\tau^2} \int_{-f_h}^{f_h} \frac{df}{|f|^{2\lambda}} (\sin(\pi f\tau))^6. \quad (13)$$

Writing this as a single-sided integral in terms of the spectral density

$$\sigma_{Ho}^2(\tau) = \frac{8}{3\pi^2\tau^2} \int_0^{f_h} \frac{S_y(f)df}{f^2} (\sin(\pi f\tau))^6. \quad (14)$$

Returning to the discussion of (12), for convenience, we introduce the abbreviation

$$K = \frac{8h_\alpha}{3\pi^2\tau^2 N \tau_0}. \quad (15)$$

If we write the sum in terms of positive frequencies only, a factor of 2 comes in except for the most positive frequency and so

$$\sigma_{Ho}^2(\tau) = K \left( \sum_{m>0}^{N/2-1} \frac{(\sin(\frac{\pi m s}{N}))^6}{f_m^{2\lambda}} + \frac{(\sin(\frac{\pi s}{2}))^6}{2(f_{N/2})^{2\lambda}} \right). \quad (16)$$

The influence of the second term in (16) is very small except when  $s \ll N$ ; we shall, therefore, neglect it in the remainder of this paper.

Similar arguments lead to known expressions for the nonoverlapping version of the Hadamard variance and the modified Hadamard variance. Proofs of these statements can be given provided no windowing or undersampling is applied to the time series. These forms of the variance will be discussed in later sections.

### IV. LIMIT OF LARGE SAMPLING TIMES

We evaluate the integral in (14) for each of the common power-law noises in the limit of large sampling times  $\tau$ . The

TABLE I  
ASYMPTOTIC EXPRESSIONS FOR THE HADAMARD AND MODIFIED  
HADAMARD VARIANCES IN THE LIMIT OF LARGE SAMPLING  
TIMES  $\tau = s\tau_0$

$S_y(f)$	$\sigma_{Ho}^2(\tau)$	Mod $\sigma_{Ho}^2(\tau)$	$\lambda$
$h_2 f^2$	$\frac{5h_2 f_h}{6\pi^2 \tau^2}$	$\frac{5h_2}{12\pi^2 \tau^3}$	0
$h_1 f$	$\frac{h_1}{6\pi^2 \tau^2} (5\gamma + 5 \ln(f_h \pi \tau) + \frac{\ln(48)}{2})$	$\frac{3h_1 \ln(\frac{4}{3})}{\pi^2 \tau^2}$	$\frac{1}{2}$
$h_0$	$\frac{h_0}{2\tau}$	$\frac{2h_0}{9\tau}$	1
$\frac{h_{-1}}{f}$	$\frac{1}{2} h_{-1} \ln\left(\frac{256}{27}\right)$	$\frac{8}{3} h_{-1} \ln\left(\frac{27 \times 3^{3/8}}{32}\right)$	$\frac{3}{2}$
$\frac{h_{-2}}{f^2}$	$\frac{1}{3} h_{-2} \pi^2 \tau$	$\frac{2\pi^2}{9} h_{-2} \tau$	2

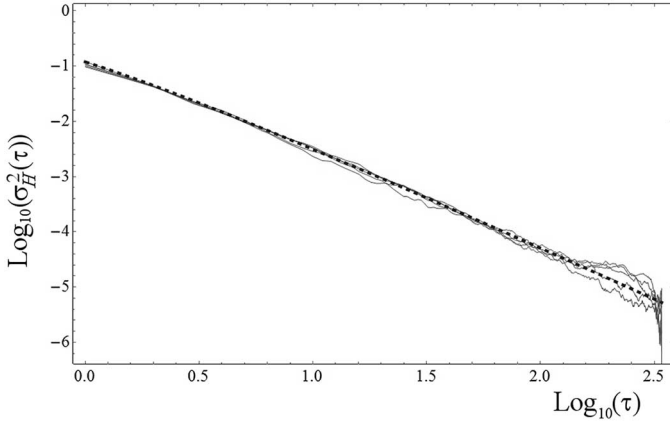


Fig. 1. Overlapping Hadamard variance for  $N = 1024$ , for flicker PM.  $h_1$  has been set equal to unity and  $\tau_0 = 1$ . Several actual simulation runs are plotted; the theoretical average is shown as a dashed line.

spectral density is  $S_y(f) = h_\alpha f^\alpha$ . Contributions to the integrals typically have oscillating terms that become small in this limit. Table I lists the limiting values of the overlapping Hadamard variance with such oscillating terms omitted.  $\gamma$  is the Euler's constant in  $\sigma_{Ho}^2(\tau)$  for  $\alpha = 1$ .

As is the case for the overlapping Allan variance, the difference between white PM and flicker PM cannot be distinguished by the dependence on  $\tau$ . Fig. 1 plots the overlapping Hadamard variance for flicker PM,  $N = 1024$ . Several simulation runs are shown as well as the average variance for  $S_y(f) = h_1 f$ .

## V. NONOVERLAPPING HADAMARD VARIANCE

In the completely overlapping form, the sum in (10) runs from 1 to  $N - 3s$ ; thus, some values of the time series would be used more than once. In order to avoid this problem, a nonoverlapping form is defined as in (10) but with the sum over  $j$  skipping repeated data items. Consider the sum

$$\frac{1}{s} \sum_{l=j}^{j+s-1} \left( \Delta_{l,s}^{(3)} \right)^2. \quad (17)$$

The sum has  $s$  terms and uses each data item from  $j$  to  $j + 4s - 1$  exactly once. This block of data is of length  $4s$ . In general,  $4s$  is incommensurable with  $N$  so there will exist some data items that will not be included in such blocks. Let  $M_{\max}$  be the maximum integer such that

$$(4M_{\max} + 1)s \leq N. \quad (18)$$

Then, if we define the nonoverlapping form of the Hadamard variance as

$$\sigma_{Hno}(\tau)^2 = \frac{1}{(M_{\max} + 1)s^2} \left\langle \sum_{M=0}^{M_{\max}} \sum_{l=1+4Ms}^{(4M+1)s} \left( \Delta_{l,s}^{(3)} \right)^2 \right\rangle \quad (19)$$

we will have left out part of one block of data, but each item of data in the sum will appear with equal weight in (19). Although an estimate of the variance could be improved by incorporating the partial block of data and appropriately modifying the normalization constant, we shall develop the theory ignoring such contributions because the ensemble average of each term in (19) is independent of  $l$ .

Squaring the third difference in (11), we write one of the factors as a complex conjugate and average over the random numbers. Then, the only terms that contribute to the double sum are those for equal frequencies, and

$$\left\langle \left( \Delta_{k,s}^{(3)} \right)^2 \right\rangle = \frac{8}{6} \frac{h_\alpha}{\pi^2 \tau^2 N \tau_0} \sum_{m>0} \frac{(\sin \frac{\pi m s}{N})^6}{|f_m|^{2\lambda}}. \quad (20)$$

Thus, the average of both overlapping and nonoverlapping forms of the Hadamard variance are the same, but as will be shown the probability distributions and confidence intervals are different.

## VI. MODIFIED OVERLAPPING HADAMARD VARIANCE

The Hadamard variance suffers from the same difficulty as does the Allan Variance—the variances for both white phase noise and flicker PM are proportional to  $\tau^{-2}$  for large  $\tau$ ; thus, the Hadamard variance cannot distinguish between white PM and flicker PM. Allan solved this problem by inventing the modified variance, which involves averaging  $s$  differences before squaring, then averaging the result. The modified Hadamard variance is defined as [23]

$$\text{Mod } \sigma_H^2(\tau) = \left\langle \left( \frac{1}{s} \sum_{l=j}^{j+s-1} \Delta_{l,s}^{(3)} \right)^2 \right\rangle \quad (21)$$

where the average is taken over the ensemble of values of the random number distributions and over all possible values of  $j$ . For the overlapping form, using elementary trigonometric identities the expression reduces to

$$\text{Mod } \sigma_H^2 = \frac{64\tau_0^4 \sigma^2}{3\tau^2 N s^2} \sum_m \frac{|f_0|^{2\lambda}}{|f_m|^{2\lambda}} \left( \sin \frac{\pi m s}{N} \right)^6 \left( \frac{\sin \frac{\pi m s}{N}}{\sin \frac{\pi m}{N}} \right)^2. \quad (22)$$

In the limit of sufficiently densely spaced frequencies, the sum passes to a single-sided integral

$$\text{Mod } \sigma_H^2(\tau) = \frac{8}{3} \int_0^{f_h} \frac{S_y(f) df}{(\pi s \tau f)^2} \frac{(\sin \pi \tau f)^8}{(\sin \pi \tau_0 f)^2}. \quad (23)$$

In general, this integral is difficult to evaluate. In Table I, we give the results in the limit

$$\tau/\tau_0 \rightarrow \infty. \quad (24)$$

As is the case for the modified Allan variance, the modified Hadamard variance distinguishes between white PM and flicker PM.

In the nonoverlapping case, the desire is to use each data item only once, but to average over  $s$  values of the third difference before squaring. The average over  $s$  values of the third difference, from (11) is

$$\begin{aligned} \frac{1}{s} \sum_{l=j}^{j+s-1} \Delta_{l,s}^{(3)} &= i \sqrt{\frac{K}{12s^2}} \sum_{l=j}^{j+s-1} \sum_m \frac{w_m}{|f_m|^\lambda} \\ &\times \left( \sin \left( \frac{\pi m s}{N} \right) \right)^3 e^{-\frac{\pi i m}{N} (2l+3s)}. \end{aligned} \quad (25)$$

In this sum, each item in the time series occurs exactly once. Therefore, as in the case of the nonoverlapping variance, the modified Hadamard variance can be constructed in terms of blocks of nonoverlapping data. Here, the sum over  $l$  is a geometric series giving

$$\begin{aligned} \frac{1}{s} \sum_{l=j}^{j+s-1} \Delta_{k,s}^{(3)} &= i \sqrt{\frac{K}{12s^2}} \sum_m \frac{w_m}{|f_m|^\lambda} \\ &\times \frac{\left( \sin \left( \frac{\pi m s}{N} \right) \right)^4}{\sin \left( \frac{\pi m}{N} \right)} e^{-\frac{\pi i m}{N} (2l+4s-1)}. \end{aligned} \quad (26)$$

Squaring and averaging, again writing one factor in terms of a complex conjugate, as in the above cases only terms of equal frequencies contribute and then summing only over positive frequencies, the modified Hadamard variance is the same as that given in (22), which can also be written

$$\text{Mod } \sigma_H^2(\tau) = \frac{8K}{3s^2} \sum_{m>0} \frac{\left( \sin \frac{\pi m s}{N} \right)^8}{|f_m|^{2\lambda} \left( \sin \frac{\pi m}{N} \right)^2}. \quad (27)$$

In Section VII, we discuss the different probability functions that arise.

## VII. EIGENVALUES AND PROBABILITIES

In this section, we shall develop expressions for the probability of observing a particular value  $A_o$  for the overlapping Hadamard variance in a single measurement, or in a single simulation run.  $A_o$  is a random variable representing a possible value of the overlapping variance. We use a subscript “ $o$ ” to denote the completely overlapping case. To save writing, we introduce the following abbreviations:

$$\begin{aligned} F_m^j &= \frac{\left( \sin \left( \frac{\pi m s}{N} \right) \right)^3}{|f_m|^\lambda} \sin \left( \frac{\pi m (2j+3s)}{N} \right) \\ G_m^j &= -\frac{\left( \sin \left( \frac{\pi m s}{N} \right) \right)^3}{|f_m|^\lambda} \cos \left( \frac{\pi m (2j+3s)}{N} \right). \end{aligned} \quad (28)$$

The dependence on  $s$  is suppressed, but is to be understood. We write the third difference in terms of a sum over positive frequencies only, keeping in mind that the most positive and the most negative frequencies only contribute a single term since  $\sin(\pi(j+s)) = 0$ . The imaginary contributions cancel, and from (11) we obtain

$$\Delta_{j,s}^{(3)} = \sqrt{K} \sum_{m>0} \left( F_m^j \frac{u_m}{\sigma} + G_m^j \frac{v_m}{\sigma} \right). \quad (29)$$

There is no term in  $v_{N/2}$ . Then from (10), the overlapping Hadamard variance is given by

$$\sigma_y^2(\tau) = \frac{K}{N-3s} \sum_{j=1}^{N-3s} \sum_{m>0} \left( (F_m^j)^2 + (G_m^j)^2 \right). \quad (30)$$

To compute the probability that a particular value,  $A_o$  is observed for the Hadamard variance, given all the possible values that the random variables  $u_1, v_1, \dots, u_{N/2}$  can have, we form the integral

$$\begin{aligned} P(A_o) &= \int \delta \left( A_o - \frac{1}{N-3s} \sum_j \left( \Delta_{j,s}^{(3)} \right)^2 \right) \\ &\times \prod_{m>0} \left( e^{-\frac{u_m^2 + v_m^2}{2\sigma^2}} \frac{du_m dv_m}{2\pi\sigma^2} \right). \end{aligned} \quad (31)$$

The Dirac delta function constrains the averaged third difference to the specific value  $A_o$  while the normally distributed random variables  $u_1, v_1, \dots, u_m, v_m, \dots, u_{N/2}$  range over their values. There is no integral for  $v_{N/2}$ . Inspecting this probability and (29) for the third difference indicates that we could dispense with the factors of  $\sigma^{-1}$  and work with normally distributed random variables having variance unity. Henceforth, we set  $\sigma = 1$ .

The exponent involving the random variables is a quadratic form that can be written in matrix form by introducing the  $N-1$  dimension column vector  $\mathbf{U}$  (the zero frequency component is excluded)

$$\mathbf{U}^T = [u_1, v_1, \dots, u_m, v_m, \dots, v_{N/2-1}, u_{N/2}]. \quad (32)$$

Then

$$\frac{1}{2} \sum_{m>0} (u_m^2 + v_m^2) = \frac{1}{2} \mathbf{U}^T \mathbf{U} = \frac{1}{2} \mathbf{U}^T \mathbf{1} \mathbf{U} \quad (33)$$

where  $\mathbf{1}$  represents the unit matrix. The delta-function in (31) can be written in exponential form by introducing one of its well-known representations, an integral over all angular frequencies  $\omega$  [14]

$$\begin{aligned} P(A_o) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega \left( A_o - \frac{1}{N-3s} \sum_j \left( \Delta_{j,s}^{(3)} \right)^2 \right)} \\ &\times \prod_{m>0} \left( e^{-\frac{u_m^2 + v_m^2}{2\sigma^2}} \frac{du_m dv_m}{2\pi\sigma^2} \right). \end{aligned} \quad (34)$$



The contour of integration goes along the real axis in the complex  $\omega$  plane.

The squared third difference is a complicated quadratic form in the random variables  $u_1, v_1, \dots, u_m, v_m, \dots, u_{N/2}$ . If this quadratic form could be diagonalized without materially changing the other quadratic terms in the exponent, then the integrals could be performed in spite of the imaginary factor  $i$  in the exponent. To accomplish this, we introduce a column vector  $\mathbf{C}^j$  that depends on  $j, m, s, N$  and whose transpose is

$$(\mathbf{C}^j)^T = [F_1^j, G_1^j, \dots, F_m^j, G_m^j, \dots, G_{N/2-1}^j, F_{N/2}^j]. \quad (35)$$

The column vector has  $N - 1$  real components. It contains all the dependence of the third difference on the frequency, averaging time  $\tau$ , and particular power-law noise. We use indices  $\{m, n\}$  as matrix (frequency) indices. The (scalar) third difference operator in (29) can be written very compactly as a matrix product

$$\Delta_{j,s}^{(3)} = \sqrt{K}(\mathbf{C}^j)^T \mathbf{U} = \sqrt{K} \mathbf{U}^T \mathbf{C}^j. \quad (36)$$

Then, the quantity to be averaged is

$$\frac{1}{N-3s} \sum_j \left( \Delta_{j,s}^{(3)} \right)^2 = \mathbf{U}^T \left( \frac{K}{N-3s} \sum_j \mathbf{C}^j (\mathbf{C}^j)^T \right) \mathbf{U}. \quad (37)$$

The matrix

$$H_o = \frac{K}{N-3s} \sum_j \mathbf{C}^j (\mathbf{C}^j)^T \quad (38)$$

is real and symmetric.  $H_o$  is also Hermitian and therefore has real eigenvalues. A real symmetric matrix can be diagonalized by an orthogonal transformation [15], [16], which we denote by  $O$ . Although we shall not need to determine this orthogonal transformation explicitly, it could be found by first finding the eigenvalues  $\epsilon$  and eigenvectors  $\psi_\epsilon$  of  $H_o$ , by solving the equation

$$H_o \psi_\epsilon = \epsilon \psi_\epsilon. \quad (39)$$

The transformation  $O$  is a matrix of dimension  $(N-1) \times (N-1)$  consisting of the components of the normalized eigenvectors placed in columns. Then

$$H_o O = O E \quad (40)$$

where  $E$  is a diagonal matrix with entries equal to the eigenvalues of the matrix  $H_o$ . Then, since the transpose of an orthogonal matrix is the inverse of the matrix

$$O^T H_o O = E. \quad (41)$$

The matrix  $H_o$  is thus diagonalized, at the cost of introducing a linear transformation of the random variables

$$\begin{aligned} \frac{1}{N-3s} \sum_j \left( \Delta_{j,s}^{(3)} \right)^2 &= \mathbf{U}^T H_o \mathbf{U} = \mathbf{U}^T O O^T H_o O O^T \mathbf{U} \\ &= (\mathbf{U}^T O) E (O^T \mathbf{U}). \end{aligned} \quad (42)$$

We introduce  $N - 1$  new random variables by means of the transformation

$$\mathbf{V} = O^T \mathbf{U}. \quad (43)$$

Then, the term in the exponent representing the Gaussian distributions is

$$\begin{aligned} -\frac{1}{2} \mathbf{U}^T \mathbf{1} \mathbf{U} &= -\frac{1}{2} \mathbf{U}^T O \mathbf{1} O^T \mathbf{U} \\ &= -\frac{1}{2} \mathbf{V}^T \mathbf{1} \mathbf{V} = -\frac{1}{2} \sum_{n=1}^{N-1} V_n^2. \end{aligned} \quad (44)$$

The Gaussian distributions remain basically unchanged.

Furthermore, the determinant of an orthogonal matrix is  $\pm 1$ , because the transpose of the matrix is also the inverse, and the total volume element for the space of random numbers is unchanged

$$dV_1 dV_2, \dots, dV_{N-1} = dU_1 dU_2, \dots, dU_{N-1}. \quad (45)$$

After completing the diagonalization

$$\frac{1}{N-3s} \sum_j \left( \Delta_{j,s}^{(3)} \right)^2 = \sum_i \epsilon_i V_i^2. \quad (46)$$

The probability is, therefore,

$$P(A_o) = \int \frac{d\omega}{2\pi} e^{i\omega(A_o - \sum_k \epsilon_k V_k^2)} \prod_i \left( e^{-\frac{V_i^2}{2}} \frac{dV_i}{\sqrt{2\pi}} \right). \quad (47)$$

An eigenvalue of zero will not contribute to this probability since the random variable corresponding to a zero eigenvalue just integrates out.

Let the eigenvalue  $\epsilon_i$  have multiplicity  $\mu_i$ , which means that the eigenvalue  $\epsilon_i$  is repeated  $\mu_i$  times. Integration over the random variables then gives a useful form for the probability

$$P(A_o) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega A_o}}{\prod_k (1 + 2i\epsilon_k \omega)^{\mu_k/2}}. \quad (48)$$

Finally, the contour integral may be deformed and closed in the upper half complex plane where it encloses the singularities of the integrand. This is discussed in detail in [1] and will not be repeated here. If  $A_o < 0$  the contour may be closed in the lower half plane where there are no singularities, so in this case  $P(A_o < 0) = 0$ .

#### A. Properties of the Eigenvalues

First, it is easily checked that the probability is correctly normalized by using properties of the delta-function

$$\begin{aligned} \int P(A_o) dA_o &= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\int e^{i\omega A_o} dA_o}{\prod_i (1 + 2i\epsilon_i \omega)^{\mu_i/2}} \\ &= \int_{-\infty}^{+\infty} \frac{\delta(\omega) d\omega}{\prod_i (1 + 2i\epsilon_i \omega)^{\mu_i/2}} \\ &= \int_{-\infty}^{+\infty} d\omega \delta(\omega) = 1. \end{aligned} \quad (49)$$

Next, let us calculate the trace of  $H_o$ . Since the trace is not changed by an orthogonal transformation

$$\begin{aligned} \text{Trace}(O^T H_o O) &= \text{Trace}(H_o O O^T) = \text{Trace}(H_o O O^{-1}) \\ &= \text{Trace}(H_o) = \sum_i \epsilon_i. \end{aligned} \quad (50)$$

The sum of the diagonal elements of  $H_o$  equals the sum of the eigenvalues of  $H_o$ . If we then explicitly evaluate the sum of the diagonal elements of  $H_o$ , we find from (50) and (28)

$$\begin{aligned} \sum_i \epsilon_i &= \frac{K}{N-3s} \sum_j \text{Trace}(\mathbf{C}^j (\mathbf{C}^j)^T) \\ &= \frac{K}{N-3s} \sum_j \sum_{m>0} \left( (F_m^j)^2 + (G_m^j)^2 \right) \\ &= K \sum_{m>0} \frac{\left( \sin \frac{\pi m s}{N} \right)^6}{|f_m|^{2-\alpha}} = \sigma_{H_o}^2(\tau). \end{aligned} \quad (51)$$

Every term labeled by  $j$  contributes the same amount. We obtain the useful result that *the overlapping Hadamard variance is equal to the sum of the eigenvalues of the matrix  $H_o$* . Similar results have been established for the various types of the Allan variance [1].

### B. Distribution of Eigenvalues

The equation for eigenvalues  $H_o \psi_\mu = \epsilon \psi_\mu$  produces many zero eigenvalues, especially when  $\tau$  is large. The dimension of the matrix  $H_o$  is, therefore, much larger than necessary. We have performed extensive numerical calculations for many different values of  $N$ , which indicate that for the completely overlapping Hadamard variance, the eigenvalue equation has a total of  $N-1$  eigenvalues, but only  $N-3s$  nonzero eigenvalues; the number of significant eigenvalues is in fact equal to the number of terms  $N-3s$  in the sum over  $j$  in the equations

$$\frac{K}{N-3s} \sum_n \sum_j^{N-3s} (\mathbf{C}_m)^j (\mathbf{C}_n)^j \psi_n = \epsilon \psi_m. \quad (52)$$

The factorized form of  $H_o$ , which arises on squaring the difference operator in (38), permits the reduction of the size of the matrix that is to be diagonalized. We introduce the quantities

$$\phi_\mu^j = \sum_n (\mathbf{C}_n)^j \psi_{n\mu}. \quad (53)$$

We use the Greek index  $\mu$  to label a nonzero eigenvalue and the index  $\nu$  to label a zero eigenvalue. The eigenvalue equation becomes

$$\frac{K}{N-3s} \sum_j (\mathbf{C}_m)^j \phi_\mu^j = \epsilon \psi_{m\mu}. \quad (54)$$

Multiply by  $(\mathbf{C}_m)^l$  and sum over the frequency index  $m$ . Then

$$\frac{K}{N-3s} \sum_{m,j} (\mathbf{C}_m)^l (\mathbf{C}_m)^j \phi_\mu^j = \epsilon \phi_\mu^l. \quad (55)$$

This is an eigenvalue equation with reduced dimension  $N-3s$  rather than  $N-1$ , since the number of possible values of  $j$  is  $N-3s$ . The eigenvalue equation can, therefore, be written in terms of a reduced matrix  $H_{\text{red}}$ , given by

$$(H_{\text{red}})^{lj} = \frac{K}{N-3s} \sum_m (\mathbf{C}_m)^l (\mathbf{C}_m)^j. \quad (56)$$

The indices  $l, j$  run from 1 to  $N-3s$ . Eigenvalues generated by (55) are all nonzero. To prove this, multiply (55) by  $\phi_\mu^l$  and sum over  $l$ . We obtain

$$\frac{K}{N-3s} \sum_m \left( \sum_l (\mathbf{C}_m)^l \phi_\mu^l \right)^2 = \epsilon \sum_l (\phi_\mu^l)^2. \quad (57)$$

The eigenvalue cannot be zero unless

$$\sum_l (\mathbf{C}_m)^l \phi_\mu^l = 0 \quad (58)$$

for every  $m$ . The number of such conditions, however, is larger than the number  $N-3s$  of variables, so the only way this can be satisfied is if  $\phi_\mu^l = 0$ , a trivial solution. Therefore to obtain normalizable eigenvectors from (55), the corresponding eigenvalues must all be positive. This is true even though some of these conditions may be trivially satisfied if the factor  $\sin(\pi m s/N)$  vanishes, which happens sometimes when

$$m s = M N \quad (59)$$

where  $M$  is an integer. Every time a solution of (59) occurs, two of (58) relating components of  $\phi_\mu^l$  are lost. Suppose there were  $n$  solutions to (59); then, the number of conditions lost would be  $2n$ . The number of variables is  $N-3s$  and the number of conditions left in (58) would be  $N-1-2n$ . The excess of conditions over variables is thus

$$N-1-2n-(N-3s) = 3s-2n-1. \quad (60)$$

It can be shown that under all circumstances  $3s-2n-1 > 0$ ; thus, the eigenvalues obtained by solving (54) are all nonzero.

We temporarily drop the subscript  $o$  since the remainder of this section is valid for any of the variances. If the eigenvalues are found and the appropriate matrix is diagonalized, we may compute the probability for observing a value of the variance, denoted by the random variable  $A$ , by

$$\begin{aligned} P(A) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(A-V^T E V)} \prod_i \left( \frac{e^{-V_i^2/2} dV_i}{\sqrt{2\pi}} \right) \\ &= \int \frac{d\omega}{2\pi} \frac{e^{i\omega A}}{\prod_i (1+2i\epsilon_i \omega)^{\mu_i/2}}. \end{aligned} \quad (61)$$

### C. Case of a Single Eigenvalue

If a single eigenvalue occurs once only, the general probability expression (61) has a single factor in the denominator, and evaluation of the integral gives

$$P(A) = \frac{1}{\sqrt{2\pi\sigma_{H_o}^2(\tau)}} \frac{e^{-A/(2\sigma_{H_o}^2(\tau))}}{\sqrt{A}}. \quad (62)$$

This is a chi-squared distribution with exactly one degree of freedom.

#### D. Case of Two Distinct Nonzero Eigenvalues

For the overlapping variance, when  $s$  has its maximum value  $N/3 - 1$  there are two unequal eigenvalues. The probability integral can be performed by closing the contour in the upper half plane and gives the expression

$$P(A) = \frac{1}{2\sqrt{\epsilon_1\epsilon_2}} e^{-\frac{A}{4}\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)} I_0\left(\frac{A}{4}\left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right)\right) \quad (63)$$

where  $I_0$  is the modified Bessel function of order zero [17], [18]. The probability is correctly normalized. It differs from a chi-squared distribution in that the density does not have a singularity at  $A = 0$ .

#### E. Case of a Single Root $\epsilon$ Repeated Three Times

The probability integral can be evaluated by integrating by parts, with the result

$$P(A) = \frac{1}{\sqrt{2\pi\epsilon^3}} A e^{-A/2\epsilon}. \quad (64)$$

Evaluation of the contour integral when there are more than two distinct eigenvalues is discussed in [1]. If an eigenvalue occurs an even number  $2n$  times, the corresponding singularity becomes a pole of order  $n$  and a chi-squared probability distribution may result; this has only been observed to occur for white PM.

#### F. Case of four Distinct Eigenvalues

For flicker PM with  $N = 1024$ , when  $\tau$  has its maximum value  $341\tau_0$ , there is only one eigenvalue. For  $\tau = 340\tau_0$ , there are four distinct eigenvalues,  $\{3.906492 \times 10^{-6}, 5.941771 \times 10^{-7}, 3.344254 \times 10^{-7}, 2.290869 \times 10^{-7}\}$ . In Fig. 2, we plot a histogram of the values of the Hadamard variance for flicker PM for the case  $N = 1024$ ,  $\tau_0 = 1$  s,  $\tau = 340$  s for the overlapping case. A histogram of the results of 5000 independent runs of the noise simulation process is also shown. Evaluation of the contour integral was discussed in detail in [1]. The probability density is given by

$$P(A) = \frac{1}{\pi} \int_{r_1}^{r_2} \frac{\sqrt{r_1 r_2 r_3 r_4} e^{-yA} dy}{\sqrt{(y-r_1)(r_2-y)(r_3-y)(r_4-y)}} - \frac{1}{\pi} \int_{r_3}^{r_4} \frac{\sqrt{r_1 r_2 r_3 r_4} e^{-yA} dy}{\sqrt{(y-r_1)(y-r_2)(y-r_3)(r_4-y)}} \quad (65)$$

where  $r_i = 1/(2\epsilon_i)$ .

For this simulation, the average variance is  $\tau = 5.046 \times 10^{-6}$  while the sum of eigenvalues is  $5.064 \times 10^{-6}$ ; the confidence limits within which there is a 50% probability of finding the variance are  $1.484 \times 10^{-6}$  and  $6.461 \times 10^{-6}$  and the median is  $3.111 \times 10^{-6}$ . For comparison, a chi-squared probability distribution with three degrees of freedom with the same average has a 50% probability of finding the variance between  $2.05 \times 10^{-6}$  and  $6.93 \times 10^{-6}$ .

For values of  $s$  that are small compared to  $N$ , matrices such as  $H_{\text{red}}$  are large and the computation of eigenvalues is

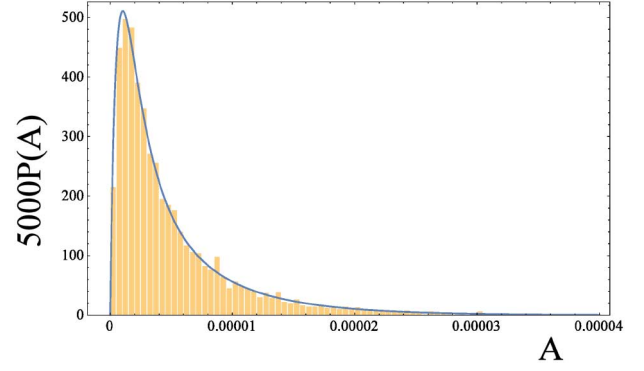


Fig. 2. Probability Density for  $N = 1024$ ,  $\tau = 340\tau_0$ , for flicker PM.  $h_1$  has been set equal to unity and  $\tau_0 = 1$ , normalized to correspond to 5000 simulation runs. A histogram of the values obtained in the independent simulation runs is shown with 80 bins in the range from 0 to 0.00004

very time-consuming. Alternatively, noisy time series generated using (3) will yield noisy values of variance from (8) and (9); a sufficient number of repetitions will then yield a distribution of values of the variance that will approach the desired probability distribution. For example with  $N = 1024$ ,  $s = 128$  the matrix  $H_o$  is of order  $640 \times 640$ , which takes a very long time to diagonalize. On the other hand (for  $h_\alpha = 1$ ), after 5000 runs the 25%, 50%, 75% levels of the cumulative distribution are 0.00002711, 0.00003119, 0.00003616, respectively, and the average variance is 0.00003237. The value of the integral (13) obtained is 0.00003230. This method was used to obtain the histogram in Fig. 2.

#### VIII. MODIFIED NONOVERLAPPING HADAMARD VARIANCE

The usual form of the Hadamard variance does not distinguish between white PM and flicker PM. These noise types are distinguished by the modified variance defined by first averaging third differences over  $s$  consecutive values and then performing the remaining averages over the ensemble of random numbers. The modified Hadamard variance is defined as

$$\text{Mod } \sigma_H^2(\tau) = \left\langle \left( \frac{1}{s} \sum_{l=j}^{j+s-1} \Delta_{l,s}^{(3)} \right)^2 \right\rangle. \quad (66)$$

The sum over  $l$  in (66) utilizes a block of data corresponding to indices from  $l = j$  to  $l = j + 4s - 1$ , with each data item included exactly once. The next block of data to be averaged would include data labeled from  $j + 4s$  to  $j + 8s - 1$ , and in general from  $j + 4Ms$  to  $j + (4M + 1)s - 1$ , where  $M$  is an integer. Starting from  $j = 1$ , there will be some maximum value of  $M$  such that

$$s(4M_{\text{max}} + 1) \leq N. \quad (67)$$

If the equality is satisfied, the data stream consists of complete blocks; if not, there will be a partial block for which the sum over  $s$  cannot be completed; we then discard the data from such an incomplete block and work only with complete blocks, for which each data item is included exactly once.

Consider a single block, starting from  $j$  as in (66) above. Then, it is straightforward to show that

$$\frac{1}{s} \sum_{l=j}^{l+s-1} \Delta_{j,s}^{(3)} = \frac{i}{s} \sqrt{K_1} \sum_m \frac{w_m}{|f_m|^\lambda} \frac{(\sin \frac{\pi m s}{N})^4}{\sin \frac{\pi m}{N}} \times e^{-\frac{i\pi m(2j+4s-1)}{N}} \quad (68)$$

where

$$K_1 = \frac{2h_\alpha}{3\pi^2\tau^2 N\tau_0} \quad (69)$$

and we have set  $\sigma = 1$ . Averaging the square of this quantity over the random variables, using (6), then writing the sum over positive frequencies, we obtain

$$\left\langle \left( \frac{1}{s} \sum_{l=j}^{j+s-1} \Delta_{j,s}^{(3)} \right)^2 \right\rangle = \frac{4K}{3s^2} \sum_{m>0} \frac{1}{|f_m|^{2\lambda}} \frac{(\sin \frac{\pi m s}{N})^8}{(\sin \frac{\pi m}{N})^2} \quad (70)$$

which is the same as for the overlapping case. If there are  $M$  blocks of nonoverlapping data in the average, then the result will still be given by (70) since the average entails division by  $M$ . If the spacing of frequencies is dense enough to pass to an integral, we obtain

$$\text{Mod } \sigma_H^2(\tau) = \frac{8}{3\pi^2\tau^2} \int_0^{f_h} df \frac{S_y(f)}{f^2} \frac{(\sin(\pi f\tau))^8}{(s \sin(\pi f\tau_0))^2}. \quad (71)$$

To derive expressions for the probability of observing a particular value of the Hadamard variance, we replace  $j$  by  $1 + 4Ms$  and write the sum in (66) over positive frequencies before averaging

$$\frac{1}{s} \sum_{l=1+4Ms}^{4Ms+s} \Delta_{l,s}^{(3)} = \frac{\sqrt{4K_1}}{s} \sum_{m>0} (F_m u_m + G_m v_m). \quad (72)$$

where

$$F_m^M = \frac{(\sin \frac{\pi m s}{N})^4}{|f_m|^\lambda \sin \frac{\pi m}{N}} \sin q \quad (73)$$

$$G_m^M = -\frac{(\sin \frac{\pi m s}{N})^4}{|f_m|^\lambda \sin \frac{\pi m}{N}} \cos q \quad (74)$$

and where  $q = \pi m(1 + 8Ms + 4s)/N$ .

With these new definitions for  $F_m^M$  and  $G_m^M$ , we define a new vector  $\mathbf{C}^M$  such that

$$(\mathbf{C}^M)^T = \{F_1^M, G_1^M, F_2^M, G_2^M, \dots, F_{N/2}^M\}. \quad (75)$$

Using (32) for the random numbers leads to

$$\frac{1}{s} \sum_{l=1+4Ms}^{4Ms+s} \Delta_{l,s}^{(3)} = \sqrt{\frac{4K_1}{s^2}} \mathbf{C}^T \mathbf{U} = \sqrt{\frac{4K_1}{s^2}} \mathbf{U} \mathbf{C}^T. \quad (76)$$

Then, the quantity to be averaged is

$$\frac{1}{M_{\max} + 1} \sum_M \left( \frac{1}{s} \sum_{l=1+4Ms}^{4Ms+s} \Delta_{j,s}^{(3)} \right)^2 = \frac{4K_1}{s^2(M_{\max} + 1)} \sum_M \mathbf{U}^T \mathbf{C}^M (\mathbf{C}^M)^T \mathbf{U}. \quad (77)$$

Defining the matrix

$$(H_H)_{mn} = \left( \frac{4K_1}{s^2(M_{\max} + 1)} \sum_M \mathbf{C}^M (\mathbf{C}^M)^T \right)_{mn} \quad (78)$$

the probability of observing a value  $A$  of the variance will be

$$P(A) = \int \frac{d\omega}{2\pi} e^{i\omega(A - \mathbf{U}^T H_H \mathbf{U}/s^2)} \times \prod_{m>0} \left( e^{-\frac{u_m^2 + v_m^2}{2\sigma^2}} \frac{du_m dv_m}{2\pi\sigma^2} \right). \quad (79)$$

Diagonalization of the matrix  $H_H$  leads in the usual way to the expression for the probability in terms of the eigenvalues  $\epsilon_i$

$$P(A) = \int_0^{f_h} \frac{d\omega}{2\pi} \frac{e^{i\omega A}}{\prod_i \sqrt{1 + 2i\omega\epsilon_i}}. \quad (80)$$

The eigenvalue equation will be

$$H_H \psi^{(\epsilon)} = \epsilon \psi^{(\epsilon)} \quad (81)$$

or

$$\frac{4K_1}{s^2(M_{\max} + 1)} \sum_{j,M} \mathbf{C}_i^M (\mathbf{C}_j^M)^T \psi_j^{(\epsilon)} = \epsilon \psi_i^{(\epsilon)}. \quad (82)$$

The number of eigenvalues can be investigated by reducing the order of the matrix. Let

$$\phi^{M(\epsilon)} = \sum_n \mathbf{C}_j^M \psi_j^{(\epsilon)}. \quad (83)$$

Then, (81) becomes

$$\frac{4K_1}{s^2(M_{\max} + 1)} \sum_M \mathbf{C}_i^M \phi^{M(\epsilon)} = \epsilon \psi_i^{(\epsilon)}. \quad (84)$$

Multiply by  $\mathbf{C}_i^L$  and sum over the frequency index. This gives

$$\sum_M (H_{\text{red}})^{LM} \phi^{M(\epsilon)} = \epsilon \phi^{L(\epsilon)} \quad (85)$$

where the reduced matrix has dimension  $M_{\max} + 1$  and is given by

$$(H_{\text{red}})^{LM} = \frac{4K_1}{s^2(M_{\max} + 1)} \sum_i \mathbf{C}_i^L \mathbf{C}_i^M. \quad (86)$$

Multiply (85) by  $\phi^{L\epsilon}$  and sum over  $L$ . The result is

$$\frac{4K_1}{s^2(M_{\max} + 1)} = \epsilon \sum_L (\phi^{L\epsilon})^2. \quad (87)$$

## IX A RADAR VARIANCE

The analysis methods developed in this paper can be extended to other variances. For example, these methods can be applied to cases in which there is dead time between measurements of average frequency during the sampling intervals. Suppose, e.g., that the measurements consist of intervals of



length  $\tau = s\tau_0$  during which an average frequency is measured, separated by dead time intervals of length  $D - \tau$  during which no measurements are available, with the possibility of significant drift during the dead times. Let the index  $j$  label the measurement intervals with  $j = 1, 2, \dots, N$ , and let  $D = d\tau_0$  with  $d$  an integer. A variance can be defined in terms of the difference between the average frequency in the  $j$ th interval and that in the interval labeled by  $j + r$

$$\Delta_{j,r,s}^{(2)} = \frac{1}{\sqrt{2}} (\bar{y}_{j+r,s} - \bar{y}_{j,s}) \quad (88)$$

where  $\bar{y}_{j,s}$  is the average frequency in the interval  $j$  of length  $s\tau_0$ . The average fractional frequency during the measurement interval  $\tau$  is

$$\bar{y}_{r,d,s} = \frac{1}{\tau} (X_{rd+s} - X_{rd}). \quad (89)$$

To eliminate drift during the dead time, a second difference of frequencies can be used

$$\Delta_{j,r,d,s}^{(3)} = \frac{1}{\sqrt{6}} (\bar{y}_{j+2r,d,s} - 2\bar{y}_{j+r,d,s} + \bar{y}_{j,d,s}). \quad (90)$$

This is a third difference in the times. Using trigonometric identities, it can be reduced to

$$\Delta_{j,r,d,s}^{(3)} = 8i\sqrt{K_1} \sum_m \frac{w_m}{|f_m|^\lambda} \sin \frac{\pi m s}{N} \times \left( \sin \frac{\pi m r d}{N} \right)^2 e^{-\pi i m (s+2jd+2rd)/N}. \quad (91)$$

Then, an appropriate variance can be defined as

$$\Psi(\tau, D)^2 = \left\langle \left( \Delta_{j,r,d,s}^{(3)} \right)^2 \right\rangle. \quad (92)$$

Performing the average and writing the result in terms of a sum over positive frequencies

$$\Psi(\tau, D)^2 = \frac{4h_\alpha}{3\pi^2\tau^2(N\tau_0)} \times \sum_{m>0} \frac{1}{|f_m|^{2\lambda}} \left( \sin \frac{\pi m s}{N} \right)^2 \left( \sin \frac{\pi m r d}{N} \right)^4. \quad (93)$$

If the measurements are sufficiently densely spaced that it is possible to pass to an integral, this can be shown to reduce to

$$\Psi(\tau, D)^2 = \frac{8}{3} \int_0^{f_h} df \frac{S_y(f)}{(\pi f \tau)^2} (\sin(\pi f r D))^4 (\sin(\pi f \tau))^2. \quad (94)$$

When  $D = \tau$  and  $r = 1$ , there is no real dead time and this variance reduces to the ordinary Hadamard variance.

## X. SUMMARY AND CONCLUSION

This paper extends the formalism, developed in [1] for simulating time series, to time series having frequency drifts such

as are common, e.g., in rubidium standards. The methods are applied to computing Hadamard variances that are used to characterize stability of clocks with drifts. These include overlapping and nonoverlapping forms of the Hadamard variance, and the modified Hadamard variance. Just as in the case of Allan variance discussed in [1], diagonalization of quadratic forms for the average variances leads to expressions for the probabilities of observing particular values of the variance for a given sampling time  $\tau = s\tau_0$ . The probabilities are expressed in terms of integrals depending on the eigenvalues of matrices formed from squares of the third differences that are used to define the Hadamard variances. The eigenvalues are usually distinct; only for white PM have eigenvalues been observed to occur (after much calculation) with multiplicities other than unity, based on our simulations. Generally speaking, the number of eigenvalues is equal to the number of terms occurring in the sum used to define averages of the third-difference operator.

It is well known that a chi-squared distribution with  $n$  degrees of freedom occurs for a variable that is the sum of squares of  $n$  normally distributed random variables. It has been shown that, with this method, probabilities for the Hadamard variance, using the present simulation method, are not always—in fact are rarely—chi-squared distributions. This is because the frequency dependence, inserted to make the time series obey the chosen noise power law, disrupts the distribution of eigenvalues in most cases.

The present approach respects all the standard expressions for spectral density and the relationships between Hadamard variance and spectral density for the common power-law noises. It also yields reasonable simulation results for power-law noises that diverge more rapidly than flicker noise at low frequencies.

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