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A General Mechanical Model for $|f|^\alpha$ Spectral Density Random Noise with Special Reference to Flicker Noise $1/|f|$

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***Abstract*—Any class of reasonable time-dependent perturbations occurring at random, under certain internal constraints, generates random noise having a spectral density varying as $|f|^\alpha$ over an arbitrarily large range of spectral frequency f only for $-2 \leq \alpha \leq 0$. A class is the set of all perturbations which are equivalent under some individual independent scaling of amplitude, scaling of time, and translation of time. A subclass is characterized by $P(\tau)$ and $A^2(\tau)$. $P(\tau)$ is the lifetime probability density. $A^2(\tau)$ is a mean square amplitude of perturbations having lifetime τ. For a given class, $|f|^{\alpha_\infty}$ and $|f|^{\alpha_0}$ are the frequency-smoothed laws in the limits of infinite and zero frequencies, respectively. Any reasonable perturbation has $\alpha_\infty \leq -2$ and $\alpha_0 \geq 0$. To generate random noise having an $|f|^\alpha$ law over an arbitrarily large range of f from a subclass chosen from any class characterized by α_∞ and α_0, it is necessary that $\alpha_\infty \leq \alpha \leq \alpha_0$. For $\alpha_\infty < \alpha < \alpha_0$, it is necessary and sufficient that such subclasses satisfy the condition, $P(\tau)A^2(\tau) \approx B\tau^{-\alpha-3}$ with B constant, over a suitable range of τ, and that $P(\tau)A^2(\tau)$ not be larger than $B\tau^{-\alpha-3}$ outside the range. This general mechanical model is of immediate value in the formulation and criticism of specific physical models of $|f|^\alpha$ noise, including flicker noise, and in computer simulation of $|f|^\alpha$ noise.**

I. INTRODUCTION

A. The Importance of Flicker Noise in Precise Experiments

IN THE DESIGN of a precision measurement or of a standard for measurement, one of the aims is to minimize the random noise of the system. It is often found that the random fluctuations are dominated by the lowest frequency portion of the noise modulation spectrum. The spectral density of this noise is observed to increase without limit as its frequency decreases. A typical behavior is as $|f|^\alpha$, with α constant and equal to -1.0, but α varies from system to system, and it is not always constant; values of -0.8 to -1.3 are common. In many cases this behavior is found to hold over many decades of frequency.[1]-[8] In this paper the term "flicker noise" will be used in a strict sense for the prototype law $|f|^{-1}$ and in a loose sense for the general behavior described above. When it was discovered in vacuum tubes[9],[10] this curious phenomenon was called "flicker effect." Different names found in the literature for similar spectral behavior in other devices are "$1/f$ noise," "excess noise," "semiconductor noise," "low-frequency noise," "contact noise," and "pink noise."

A measurement with results which are limited by random noise with an $|f|^{-1}$ law has a fixed precision which cannot be improved by increasing the length of each run.[11],[12] In contrast, for a measurement limited by white noise (spectral density law of f^0) any desired precision may be attained by increasing the duration τ_e of the experiments; the improvement accrues as $\sqrt{\tau_e}$, as, for example, when thermal agitation is the limiting noise. The spot noise factor[13] of a device displaying flicker noise is necessarily appreciably greater than unity below some particular modulation frequency and becomes progressively worse toward lower frequencies.

For an example, consider the stability of quartz crystal oscillators, which are widely used as secondary frequency standards. Even the best quartz crystal oscillators are

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limited by their flicker frequency fluctuations to have an imprecision of about three parts in 10^{13} or worse.[14]–[18] No improvement beyond the three parts in 10^{13} is possible by utilizing longer averaging times. If it were not for the flicker noise, the precision of ten-second-interval measurements of frequency would be better than one part in 10^{14}. Such precision *is needed* by the author in a slave oscillator to be used with the atomic hydrogen maser frequency standard.

Flicker noise is ubiquitous. It is found not only in quartz crystal oscillators, but also in such things as vacuum tubes, field effect and bipolar transistors, semiconductor diodes, resistors, thermistors, carbon microphones, thin films, and light sources.[1]–[9],[11],[14] The fluctuations of a membrane potential in a biological system have recently been reported to have a flicker noise spectrum.[19],[20] No electronic low-frequency amplifier has been found to be free of flicker noise. Yet in most devices its cause is not understood. It may be presumed that if the cause were understood, then the flicker noise could be reduced. It is hoped that the model presented here will help in the understanding and ultimate control of flicker noise.

Although the case of $\alpha \approx -1$ is common, there are some interesting processes found in nature where α is much more negative than -1. For example, the fluctuations of the frequency of rotation of the earth[21] are described by $\alpha \approx -2$, and the power spectral density of galactic radiation noise[22] is described by $\alpha \approx -2.7$. The consequences of the low-frequency divergence of fluctuation amplitude when $\alpha \leq -1$ are often not easily comprehended by someone accustomed only to white noise spectral density. The mechanical model presented here can help one comprehend the divergence and its consequences. The model is applicable to the noise law $|f|^\alpha$ for any value of α.

B. *The Problem of Generating $|f|^\alpha$ Noise*

Since flicker noise or other low-frequency divergent noise is often encountered in measurements, it is useful to be able to fashion models which can generate noise having the appropriate spectral density. The models should be physically plausible, and preferably they should be chosen to correspond as closely as possible to the physical process being measured. This paper describes a rather general model which can generate noise with a spectral density of $|f|^\alpha$, any α, for a very broad range of physically plausible processes.

We consider a system in which some process of interest is occurring, and in which the process is suffering many time-dependent perturbations in a random manner due to some unspecified agent or agents. The uniform portion of the process constitutes a signal, and the random fluctuations are the associated noise. We wish to answer at least two questions.

1) Can such a superposition of time-dependent perturbations of a specified shape, occurring at random, give rise to an $|f|^\alpha$ dependence of the noise spectral density with α constant over an arbitrarily large range of spectral frequency, f?

2) What conditions must be satisfied to obtain the law $|f|^\alpha$?

To answer these questions, the discussion needs to be concerned only with the mathematical description of the time-dependent perturbations. Hence there is no reference to specific *physical models*. To avoid the possibility of the model thereby synthesized being nonphysical in principle, some constraints are imposed, e.g., certain infinities are excluded from consideration. We use the term *mechanical model* for a model which is physical in principle, although not necessarily encountered in reality.

II. Synthesis of a Mechanical Model of $|f|^\alpha$ Noise

A. *Definitions*

Let H be the representation in the time domain (t) of a perturbation of the class $\mathscr{H}$. The representation $H(a, \tau)$ has lifetime τ and squared amplitude a^2. These quantities will be defined in this part of Section II. A class is the set of all perturbations which are equivalent under some individual independent scaling of amplitude a, scaling of time t, and translation of time. Special consideration is given to reasonable perturbations. By definition a *reasonable perturbation* 1) is everywhere finite, 2) has a finite time integral, 3) has finite, nonzero energy, 4) has finite, nonzero lifetime, and 5) is a real function of time t. These may be stated symbolically, as follows.

$$-\infty < H < \infty, \quad \text{all} \quad t, \tag{1a}$$

$$-\infty < \int_{-\infty}^{+\infty} H dt < \infty, \tag{1b}$$

$$0 < \int_{-\infty}^{+\infty} H^2 dt < \infty, \tag{1c}$$

$$0 < \tau < \infty, \tag{1d}$$

and

$$H = H^* \equiv \text{complex conjugate of } H. \tag{1e}$$

Throughout the following, the time t need not appear explicitly as an argument of H, and it will not be shown. Under scaling of amplitude, the representation has the property

$$a_0 H(a, \tau) = a H(a_0, \tau). \tag{2}$$

Let $F(\omega, \tau)$ denote the Fourier transform[23] of $a^{-1}H(a, \tau)$,

$$F(\omega, \tau) \equiv a^{-1} \int_{-\infty}^{+\infty} H(a, \tau) e^{-i\omega t} dt = F^*(-\omega, \tau), \tag{3}$$

$$\omega \equiv 2\pi f, \tag{4}$$

where f is cyclic frequency. Under scaling of frequency and time, the Fourier transform has the property

$$\tau_0 F(\omega, \tau) = \tau F(\omega\tau/\tau_0, \tau_0). \tag{5}$$

The energy q of a perturbation is defined as

$$q(a, \tau) \equiv \int_{-\infty}^{+\infty} H^2(a, \tau) dt = a^2 \int_{-\infty}^{+\infty} |F(\omega, \tau)|^2 df. \tag{6}$$

Under scaling, q has the property

$$a_0^2\tau_0 q(a,\tau) = a^2\tau_0 q(a_0,\tau) = a^2\tau q(a_0,\tau_0). \tag{7}$$

The lifetime τ of a perturbation is defined implicitly by

$$\int_{-1/(2\pi\tau)}^{+1/(2\pi\tau)} |F(\omega,\tau)|^2 df \equiv \frac{1}{2}\int_{-\infty}^{+\infty} |F(\omega,\tau)|^2 df. \tag{8}$$

This definition of τ is chosen so that $|\omega|_m = 2\pi|f|_m = \tau^{-1}$, where $|f|_m$ is the median absolute frequency of the energy distribution. The utility of this definition may be seen later (in Section II-B-C) when we consider the frequency range over which a particular noise law is valid.

The square of the amplitude is defined as

$$a^2 \equiv \frac{q(a,\tau)}{\tau}. \tag{9}$$

For simplicity we set a_0^2 and τ_0 equal to unity. Hence $q(a_0, \tau_0)$ equals unity, and we note

$$q(a,\tau) = a^2\tau, \tag{10}$$

$$\int_{-\infty}^{+\infty} |F(x,\tau_0)|^2 dx = 2\pi\tau_0 = 2\pi. \tag{11}$$

Suppose a process is perturbed by a large number of time-dependent perturbations which form a subclass of the class $\mathscr{H}$. We define $A^2(\tau)$ as the mean square amplitude of perturbations in the subclass having the same lifetime τ, and we define $P(\tau)$ as the normalized probability of occurrence in the subclass of perturbations of lifetime τ. We further suppose the perturbations occur randomly (as a Poisson process) with the finite average total rate of R perturbations per unit interval of time. Hence the rate at which perturbations of lifetime τ occur is given by the product $RP(\tau)$.

The definitions given above establish a mathematical description of a general mechanical model for the generation of random noise. Using these concepts in the situation described in the preceding paragraph, the equation for the two-sided spectral density $S(f)$ of the noise is defined as

$$S(f) \equiv R\int_0^{+\infty} P(\tau)A^2(\tau)|F(\omega,\tau)|^2 d\tau. \tag{12}$$

This definition of the spectral density is consistent with that of Blackman and Tukey.[23] The dimensionality of $S(f)$ is the same as the dimensionality of the ratio H^2/f. For pure real perturbations, $S(f)$ is symmetrical about zero frequency.

Appendix I illustrates by mathematical example some of the above concepts. Fig. 1 shows two members of the specific class of reasonable perturbations discussed in Appendix I. Fig. 2 shows the corresponding squared Fourier transforms.

B. Generation of Flicker Noise, $\alpha = -1$

Equation (12) for the spectral density is put in the form

$$S(f) = \frac{R}{|\omega|}\int_0^{+\infty} [P(\tau)A^2(\tau)][\tau/\tau_0]^2 |F(\omega\tau/\tau_0, \tau_0)|^2 d(|\omega|\tau). \tag{13}$$

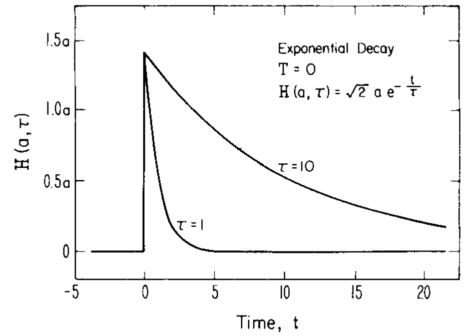

Fig. 1. Two reasonable perturbations from the class having, in the time domain, a step change followed by exponential decay back to the origin. Two different lifetimes are shown (time domain). See Appendix I.

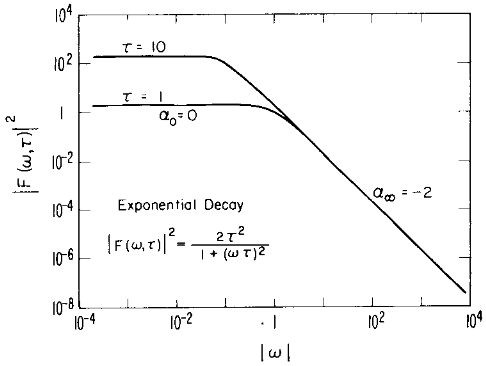

Fig. 2. The square of the normalized Fourier transform of the two reasonable perturbations of Fig. 1 are plotted against ω (frequency domain). See Appendix I.

The dependence on τ of $P(\tau)A^2(\tau)$ is expressed as

$$P(\tau)A^2(\tau) \approx B\tau^{-\delta-3}. \tag{14}$$

In general, B and δ may be functions of τ, but our interest will focus on those situations in which B and δ are independent of τ over some finite range. Consider the case where, in the range $\varepsilon \le \tau \le \zeta$, $\varepsilon > 0$,

$$\delta = -1,$$
$$B = \text{constant},$$
$$P(\tau)A^2(\tau) \approx B\tau^{-2},$$

and $P(\tau)A^2(\tau) \le B\tau^{-2}$ for τ outside of the range ε to ζ. For the spectral range $\zeta^{-1} \ll |\omega| \ll \varepsilon^{-1}$,

$$S(f) \approx \frac{RB}{|\omega|}\int_{|\omega|\varepsilon}^{|\omega|\zeta} |F(\omega\tau, 1)|^2 d(|\omega|\tau), \tag{15}$$

$$\int_{|\omega|\varepsilon}^{|\omega|\zeta} |F(x,1)|^2 dx \approx \int_0^{+\infty} |F(x,1)|^2 dx = \pi. \tag{16}$$

Hence,

$$S(f) \approx \tfrac{1}{2}RB|f|^{-1} = \tfrac{1}{2}RP(1)A^2(1)|f|^{-1}, \tag{17}$$

for frequencies in the arbitrarily large but finite range $(2\pi\zeta)^{-1} \ll |f| \ll (2\pi\varepsilon)^{-1}$.

This result shows that a spectral density law of $|f|^{-1}$ can be obtained over an arbitrarily large frequency range from a subclass *chosen from any class of reasonable perturbations.* The *sufficient* condition is that such subclass have the property $P(\tau)A^2(\tau) \approx (\text{const.})\tau^{-2}$ within a suitable range of τ, with $P(\tau)A^2(\tau) \leq (\text{const.})\tau^{-2}$ outside that range. It will be concluded in Section III that this is also the *necessary* condition for obtaining the $|f|^{-1}$ noise law from a subclass chosen from *any class* of reasonable perturbations.

C. Extension to $|f|^\alpha$ Noise, Any α

In the case of general α the starting point is (12), which is exact. Some approximations follow. Possibly the approximations are more crude than is necessary, but the author believes that they introduce no error into the conclusions. It is hoped that the simplicity of the treatment allows each conclusion and its significance to be quickly understood.

Let G be a frequency-smoothed representation of the magnitude squared of the Fourier transform of a time-dependent perturbation. Consider the behavior of G at very high and at very low frequencies, and express it as

$$\lim_{|f| \to \infty} G = (\text{const.})|f|^{\alpha_\infty}, \tag{18}$$

$$\lim_{|f| \to 0} G = (\text{const.}')|f|^{\alpha_0}. \tag{19}$$

All perturbations of the same class have the same α_∞ and the same α_0. For perturbations of finite, nonzero energy q,

$$\alpha_\infty < -1, \tag{20a}$$

$$\alpha_0 > -1. \tag{20b}$$

It has been pointed out by van der Ziel[24] that these inequalities apply to the net noise spectral density for any physical process having a bounded continuous autocovariance function. However, for any process suffering from reasonable perturbations, the exponents must satisfy the stronger constraints

$$\alpha_\infty \leq -2, \tag{21a}$$

$$\alpha_0 \geq 0. \tag{21b}$$

Appendix II has a brief discussion of α_∞ and α_0, and it includes a demonstration of the above constraints.

Consider all subclasses of reasonable perturbations for which α_∞ and α_0 are the limiting exponents as defined above. The noise spectral density of each is given by (12). We make a substitution using (14),

$$P(\tau)A^2(\tau) \approx B\tau^{-\delta-3},$$

and further limit the consideration to all such subclasses for which B and δ are approximately independent of τ over the finite range $\varepsilon \leq \tau \leq \zeta$, $\varepsilon > 0$, with the additional constraint that $P(\tau)A^2(\tau) \leq B\tau^{-\delta-3}$ for all τ outside the range ε to ζ. For the range

$$\zeta^{-1} << |\omega| << \varepsilon^{-1}, \tag{22}$$

$$S(f) \approx \frac{R}{|\omega|} \int_{|\omega|\varepsilon}^{|\omega|\zeta} B\tau^{-\delta-1}|F(\omega\tau, 1)|^2 d(|\omega|\tau), \tag{23}$$

$$S(f) \approx RB|\omega|^\delta \int_{|\omega|\varepsilon}^{|\omega|\zeta} |x|^{-\delta-1}|F(x, 1)|^2 dx. \tag{24}$$

The definite integral in (24) has the following properties:

$$\int_{|\omega|\varepsilon}^{|\omega|\zeta} x^{-\delta-1}|F(x,1)|^2 dx \approx \begin{cases} C|\omega|^{\alpha_0-\delta}, \text{ for } \delta > \alpha_0; & (25a)\\ C'|\omega|^0 = C', \text{ for } \alpha_\infty < \delta < \alpha_0; & (25b)\\ C''|\omega|^{\alpha_\infty-\delta}, \text{ for } \delta < \alpha_\infty; & (25c)\end{cases}$$

where C, C', and C'' are constants, independent of ω. Hence the frequency dependence of the noise spectral density is

$$S(f) \approx \begin{cases} (\text{const.})|f|^{\alpha_0}, & \text{for } \delta > \alpha_0; & (26a)\\ (\text{const.}')|f|^{\delta}, & \text{for } \alpha_\infty < \delta < \alpha_0; & (26b)\\ (\text{const.}'')|f|^{\alpha_\infty}, & \text{for } \delta < \alpha_\infty. & (26c)\end{cases}$$

Equations (25a–c) are obtained in Appendix III, where it is also shown that

$$C' \approx \pi \frac{(\alpha_0 + 1)(\alpha_\infty + 1)}{(\alpha_0 - \delta)(\alpha_\infty - \delta)}. \tag{27}$$

Hence, for $\alpha_\infty < \delta < \alpha_0$, the two-sided noise spectral density is

$$S(f) \approx RB\pi \frac{(\alpha_0 + 1)(\alpha_\infty + 1)}{(\alpha_0 - \delta)(\alpha_\infty - \delta)} |\omega|^\delta. \tag{28}$$

Note that for the case of $\delta = -1$, (28) is the same result as (17).

Equations (26a–c) give the interesting result that, for any smooth distribution of lifetimes of the perturbations, the exponent α of the frequency dependence of the noise spectral density cannot lie outside of the interval α_∞ to α_0 over an arbitrarily large range of frequency for all subclasses characterized by α_∞ and α_0. Equation (26b) shows that under certain conditions, which are of interest, the exponent α is equal to δ.

III. Results

We have found some interesting and useful answers to our questions. The concepts α_∞ and α_0 are important when considering the general problem of creating noise with $|f|^\alpha$ spectral density. Not all classes of reasonable perturbations characterized by α_∞ and α_0 can generate $|f|^\alpha$ noise over an arbitrarily large range of f unless there is a sufficiently broad, smooth distribution of lifetimes in the chosen subclasses. An arbitrarily smooth distribution which is arbitrarily broad gives rise to an arbitrarily smooth spectrum in which local peaks and valleys no longer exist even though they may be present in the individual $|aF(\omega, \tau)|^2$ spectra which make up the superposition. In such a superposition, at sufficiently high frequencies the spectrum varies as $|f|^{\alpha_\infty}$, and at sufficiently low frequencies the spectrum varies as $|f|^{\alpha_0}$. For any arbitrarily large range of intermediate frequencies, as the operative value of δ is varied over its domain from an arbitrarily small value to an arbitrarily large value, the only values of α which arise from such distributions lie in the interval $\alpha_\infty \leq \alpha \leq \alpha_0$. For the interval $\alpha_\infty < \alpha < \alpha_0$, α and δ have the one-to-one correspondence $\alpha = \delta$.

Some conclusions are summarized in the following statements.

1) To generate random noise having an $|f|^\alpha$ spectral density over an arbitrarily large range of f from a subclass of reasonable perturbations occurring at random, chosen from

any class characterized by α_∞ *and* α_0, it is necessary that $\alpha_\infty \le \alpha \le \alpha_0$.

2) To generate random noise having an $|f|^\alpha$ spectral density over the arbitrarily large range

$$(2\pi\zeta)^{-1} \ll |f| \ll (2\pi\varepsilon)^{-1} \tag{29}$$

from a subclass of reasonable perturbations occurring at random, chosen from *any class characterized by* α_∞ *and* α_0, for α in the more restricted interval $\alpha_\infty < \alpha < \alpha_0$, it is *necessary and sufficient* that such subclass satisfy the condition

$$P(\tau)A^2(\tau) \approx B\tau^{-\alpha-3}, \tag{30}$$

with B constant, over the range $\varepsilon \le \tau \le \zeta$, and that $P(\tau)A^2(\tau) \le B\tau^{-\alpha-3}$ for τ outside the range ε to ζ.

3) Any reasonable perturbation has $\alpha_\infty \le -2$ and $\alpha_0 \ge 0$. At least one class of reasonable perturbations exists which has $\alpha_\infty = -2$ and $\alpha_0 = 0$ (see Appendix I). Indeed, an unlimited number of such classes exist.

4) It follows from statements 1), 2), and 3) that the only values of α for which $|f|^\alpha$ noise, over an arbitrarily large range of f, can be generated by a subclass chosen from *any class* of reasonable perturbations are in the interval $-2 \le \alpha \le 0$. *This interval of* α *is centered on the flicker law*, $\alpha = -1$.

5) For the smoothed random noise spectral density due to a subclass of reasonable perturbations occurring at random, chosen from any class characterized by α_∞ and α_0, the law $|f|^{\alpha_\infty}$ holds at sufficiently high frequencies. The law $|f|^{\alpha_\infty}$ also holds for the spectral density in the frequency range $(2\pi\zeta)^{-1} \ll |f| \ll (2\pi\varepsilon)^{-1}$ if, for $\varepsilon \le \tau \le \zeta$,

$$P(\tau)A^2(\tau) \approx B\tau^{-\delta-3},$$
$$B = \text{constant},$$
$$\delta < \alpha_\infty,$$

and also if, for $\tau < \varepsilon$, $P(\tau)A^2(\tau) \le B\tau^{-\alpha_\infty-3}$. Similarly, for such a subclass, the smoothed law $|f|^{\alpha_0}$ holds at sufficiently low frequencies. The law $|f|^{\alpha_0}$ also holds for the spectral density in the frequency range $(2\pi\zeta)^{-1} \ll |f| \ll (2\pi\varepsilon)^{-1}$ if, for $\varepsilon \le \tau \le \zeta$,

$$P(\tau)A^2(\tau) \approx B\tau^{-\delta-3},$$
$$B = \text{constant},$$
$$\delta > \alpha_0,$$

and also if, for $\tau > \zeta$, $P(\tau)A^2(\tau) \le B\tau^{-\alpha_0-3}$.

It is helpful to chart the logarithm of the spectral density versus the logarithm of the frequency when considering the above results.

IV. Discussion

This general method of generating an $|f|^\alpha$ spectral density gives special attention to the factor $P(\tau)A^2(\tau)$ of (12). A second method exists, in which the special attention is given instead to the *shape* of each of the time-dependent perturbations.[25],[26] This second method is severely restricted in its choice of classes of perturbations which can generate $|f|^\alpha$ noise for any specific value of α. In this second method, a physical explanation of a flicker noise spectrum extending over more than a couple of decades of frequency requires postulating the existence of specially shaped, time-dependent perturbations which are rarely, or never, found. On the other hand, the random perturbations found to be affecting physical processes often do have broad distributions of their lifetime square-amplitude product, one of the necessary conditions for the proper application of the first method. The diffusion process is a prime example of a physical process which gives rise to broad distributions of lifetimes.[27]–[32] The first person to point out that a distribution of τ might be important appears to have been Bernamont[33] in 1937. Since then, many investigators have invoked particular distributions of τ in attempts to explain flicker noise.[1]–[3],[24],[27],[30]–[32],[34]–[37]

The effects of noise on the action of an instrument can be simulated on a computer. With this general mechanical model, noise spectral density of any desired law, or combination of laws, can be accurately and efficiently generated by a simple computer program.[38] This mechanical model, besides being general, is more efficient than a mathematical model proposed recently by Barnes and Allan[26] for simulating flicker noise with a computer.

The model as presented deals with certain portions of the total spectrum. To give an approximate description of the entire spectral domain, the model can be applied successively to each portion of the spectrum and to each operative class, as necessary.

It is apparent that the experimental observation, per se, of a noise law of $|f|^\alpha$, $-2 \le \alpha \le 0$, gives no information concerning the identity of the class (or classes) of reasonable perturbations which is operative. The corresponding statement is not true for other values of α, and the further from this range is the observed α, the more restricted is the set of classes of reasonable perturbations which might be operative.

From a physical viewpoint, the restrictions which have been placed on reasonable perturbations are not severe. Many random noise processes of physical importance may be represented by reasonable perturbations, and to them the results of this paper are applicable without modification. In some cases it is helpful to consider a time derivative or a running time integral of a process, instead of the process itself, if the perturbations involved are thereby converted into reasonable perturbations. The present results are also applicable to many random physical processes where each member of the ensemble of perturbing agents is most conveniently regarded as being of infinite liefetime,[3] provided that the principle of ergodicity can be used.

The present results are of immediate value in allowing one to evaluate and criticize proposed physical models of $|f|^\alpha$ noise. The present results also can give insight into the problem of devising models which may explain noise spectra observed in specific physical situations. For example, a new mechanism, hypothetical and untested but physically plausible, for the flicker of frequency of quartz crystals has been devised.[39] It is emphasized that, to develop a physical model for flicker noise, the crucial problem is to find the physical circumstances which cause the product

$P(\tau)A^2(\tau)$ to vary approximately as $\tau^{-\alpha-3}$; the shape or class of the perturbation is probably irrelevant. That it is the product $P(\tau)A^2(\tau)$ which is the pertinent factor seems not to have been appreciated in previous discussions of flicker noise.[1]–[3],[10],[24],[27],[33]–[37]

Appendix I

An example of a specific class will help to clarify the H representation and some of the concepts associated with it. Consider the class of all perturbations each of which is represented in the time domain by a step change followed by exponential decay back to the origin (see Figs. 1 and 2).

$$
\begin{aligned}
H(a,\tau) &= 0,\ t \le T.\\
H(a,\tau) &= \sqrt{2}ae^{-(t-T)/\tau}, t \ge T.\\
H(1,\tau) &= \sqrt{2}e^{-(t-T)/\tau}, t \ge T.\\
F(\omega,\tau) &= \sqrt{2}[\tau^{-1} + i\omega]^{-1}e^{-i\omega T}.\\
F(\omega,1) &= \sqrt{2}[1 + i\omega]^{-1}e^{-i\omega T}.\\
G &= (\text{const.})[\tau^{-2} + \omega^2]^{-1}.\\
\alpha_0 &= 0.\\
\alpha_\infty &= -2.
\end{aligned}
$$

Appendix II

For reasonable perturbations,

$$\lim_{\omega\to 0} |F(\omega,\tau)|^2 = |F(0,\tau)|^2 = \left|\int_{-\infty}^{+\infty} H(1,\tau)dt\right|^2 < \infty. \tag{31}$$

This requires the frequency-smoothed behavior, (19), which defines α_0, to have the constraint

$$\lim_{f\to 0} G = \lim_{f\to 0} (\text{const.})|f|^{\alpha_0} < \infty. \tag{32}$$

Hence, $\alpha_0 \ge 0$ for reasonable perturbations.

It follows from the square-integrability of reasonable perturbations (1c) that α_∞ must be less than -1 for all classes, while Appendix I shows that α_∞ equals -2 for at least one class. We will now show that there are no reasonable perturbations with α_∞ greater than -2.

By analogy with the Wiener-Khinchin theorem,[40] $|F(\omega, 1)|^2$ is the Fourier transform of the autocovariance function for $H(1, 1)$. If $H(1, 1)$ is a reasonable perturbation, then the autocovariance function is everywhere continuous, and the first derivative with respect to time of the autocovariance function is everywhere finite. Hence

$$\left|\frac{d}{dT}\int_{-\infty}^{+\infty}\frac{1}{2\pi}\left|F(\omega,1)\right|^2 e^{i\omega T}d\omega\right| < \infty, \tag{33}$$

$$\left|\int_0^\infty \left|F(\omega,1)\right|^2 \omega \sin \omega T d\omega\right| < \infty. \tag{34}$$

Equation (34) must hold for all T, including in the limit as T goes to zero.

We approximate the magnitude squared of the normalized Fourier transform of a reasonable perturbation with

$$|F(x,1)|^2 \approx K|x|^{\alpha_0} \tag{35}$$

for $|x| \le 1$, and for $|x| \ge 1$ we use the approximation

$$|F(x,1)|^2 \approx K|x|^{\alpha_\infty}, \tag{36}$$

where K is a constant, and $0 < K < \infty$. Equations (35) and (36) are used in the integral in (34), which is considered in three parts, and each part is evaluated in the limit as T goes to zero.

$$\lim_{T\to 0}\left|\int_0^1 \omega^{\alpha_0+2}Td\omega + \int_1^{1/T}\omega^{\alpha_\infty+2}Td\omega + \int_{\omega T=1}^{\omega T=\infty}\frac{(\omega T)^{\alpha_\infty+1}\sin \omega T d(\omega T)}{T^{\alpha_\infty+2}}\right| < \infty, \tag{37}$$

$$\lim_{T\to 0}\left|\left(\frac{1}{\alpha_0+3} - \frac{1}{\alpha_\infty+3}\right)T + \left(\frac{1}{\alpha_\infty+3} + \int_1^\infty x^{\alpha_\infty+1}\sin x dx\right)\left(\frac{1}{T}\right)^{\alpha_\infty+2}\right| < \infty, \tag{38}$$

$$\lim_{T\to 0}\left|\left(\frac{1}{T}\right)^{\alpha_\infty+2}\right| < \infty. \tag{39}$$

By (39) we see that $\alpha_\infty \le -2$ for reasonable perturbations.

It is interesting to note that $\alpha_0 = 0$ for a reasonable perturbation H, if and only if

$$\int_{-\infty}^{+\infty} H(a,\tau)dt \ne 0,$$

a condition which describes a shot noise perturbation, for example. For a reasonable perturbation H, $\alpha_\infty = -2$ if and only if H has at least one finite discontinuity. Arbitrarily large positive values of α_0 may be realized by interposing an arbitrarily large number of high-pass filters between the perturbing agents and the process of interest. Independently, arbitrarily large negative values of α_∞ may be realized by interposing an arbitrarily large number of low-pass filters between the perturbing agents and the process of interest.

Appendix III

As in Appendix II, we approximate the magnitude squared of the normalized Fourier transform of a reasonable perturbation with

$$|F(x,1)|^2 \approx K|x|^{\alpha_0} \tag{40}$$

for $|x| \le 1$, and for $|x| \ge 1$ we use the approximation

$$|F(x,1)|^2 \approx K|x|^{\alpha_\infty}. \tag{41}$$

The common coefficient K is a constant, and it is evaluated by equating

$$\int_0^1 K|x|^{\alpha_0}dx + \int_1^\infty K|x|^{\alpha_\infty}dx = \int_0^\infty |F(x,1)|^2dx = \pi. \tag{42}$$

Hence,

$$K = \pi\,\frac{(\alpha_0+1)(\alpha_\infty+1)}{(\alpha_\infty-\alpha_0)}. \tag{43}$$

Using (24), with (22) valid,

$$S(f) \approx RB|\omega|^{\delta} I(\omega), \tag{44}$$

where

$$I(\omega) \equiv \int_{|\omega|\varepsilon}^{|\omega|\zeta} |x|^{-\delta-1} |F(x,1)|^2 dx. \tag{45}$$

Using (40) and (41),

$$I(\omega) \approx K\left[\int_{|\omega|\varepsilon}^{1} |x|^{\alpha_0-\delta-1} dx + \int_{1}^{|\omega|\zeta} |x|^{\alpha_\infty-\delta-1} dx\right]. \tag{46}$$

For $\delta \neq \alpha_0$ and $\delta \neq \alpha_\infty$,

$$I(\omega) \approx K\left[\frac{1-(|\omega|\varepsilon)^{\alpha_0-\delta}}{(\alpha_0-\delta)} + \frac{(|\omega|\zeta)^{\alpha_\infty-\delta}-1}{(\alpha_\infty-\delta)}\right]. \tag{47}$$

Because we are considering only reasonable perturbations for which $\alpha_\infty < \alpha_0$, then, for the range of ω given by (22), and for $\alpha_\infty < \delta < \alpha_0$, $I(\omega)$ can be approximated by

$$I(\omega) \approx K\left[\frac{1}{(\alpha_0-\delta)} - \frac{1}{(\alpha_\infty-\delta)}\right] \approx \pi \frac{(\alpha_0+1)(\alpha_\infty+1)}{(\alpha_0-\delta)(\alpha_\infty-\delta)}. \tag{48}$$

Similarly, for $\delta > \alpha_0$,

$$I(\omega) \approx K\left[-\frac{(|\omega|\varepsilon)^{\alpha_0-\delta}}{(\alpha_0-\delta)}\right] \approx -\pi \frac{(\alpha_0+1)(\alpha_\infty+1)\varepsilon^{\alpha_0-\delta}}{(\alpha_0-\delta)(\alpha_\infty-\alpha_0)} |\omega|^{\alpha_0-\delta}. \tag{49}$$

Similarly, for $\delta < \alpha_\infty$,

$$I(\omega) \approx K\left[\frac{(|\omega|\zeta)^{\alpha_\infty-\delta}}{(\alpha_\infty-\delta)}\right] \approx \pi \frac{(\alpha_0+1)(\alpha_\infty+1)\zeta^{\alpha_\infty-\delta}}{(\alpha_\infty-\delta)(\alpha_\infty-\alpha_0)} |\omega|^{\alpha_\infty-\delta}. \tag{50}$$

For $\delta = \alpha_0$, by (46),

$$I(\omega) \approx K\left[\ln \frac{1}{|\omega|\varepsilon} + \frac{(|\omega|\zeta)^{\alpha_\infty-\alpha_0}-1}{(\alpha_\infty-\alpha_0)}\right], \tag{51}$$

$$I(\omega) \approx K \ln \frac{1}{|\omega|\varepsilon} \approx -\pi \frac{(\alpha_0+1)(\alpha_\infty+1)(\ln|\omega|\varepsilon)}{(\alpha_\infty-\alpha_0)}. \tag{52}$$

For $\delta = \alpha_\infty$, by (46),

$$I(\omega) \approx K\left[\frac{1-(|\omega|\varepsilon)^{\alpha_0-\alpha_\infty}}{(\alpha_0-\alpha_\infty)} + \ln|\omega|\zeta\right], \tag{53}$$

$$I(\omega) \approx K \ln|\omega|\zeta \approx +\pi \frac{(\alpha_0+1)(\alpha_\infty+1)(\ln|\omega|\zeta)}{(\alpha_\infty-\alpha_0)}. \tag{54}$$

The logarithmic dependence on $|\omega|$ which appears in (52) and (54) is expected. It is not a spurious consequence of the approximations.

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