TIME DEVIATION AND TIME PREDICTION ERROR FOR CLOCK SPECIFICATION, CHARACTERIZATION, AND APPLICATION

David W. Allan and Helmut Hellwig

Time & Frequency Division
National Bureau of Standards

Abstract

The characterization of oscillators and clocks in terms of frequency fluctuations as a function of averaging time, $\sigma_y(\tau)$ is now well understood and documented. However, a need still exists to describe a time deviation measure x(t) We show the dependence of such a deviation measure and the error of prediction of this measure on practical measurement constraints, on systematic characteristics, as well as on $\sigma_y(\tau)$. We develop models which allow clock characterization from insufficient data. For example, when statistically significant data are constrained to the short term region it is still possible to extrapolate into the long term.

Finally, we summarize the state-of-the-art of clocks and oscillators in terms of x and show the obvious importance of these measures to properly choosing, measuring, specifying, or using clocks and oscillators for position location or navigation applications.

Introduction

With the exception of line of sight ranging techniques, today's precise navigation relies on the availability of accurate time. The navigator typically either receives precise time information via radio signal or even has his reference clock on board his vehicle. Modern navigation systems such as Loran-C or the Global Positioning System (GPS) are or will be based on atomic clocks of very high performance at the location of the origin of the navigational signal. These origins may be ground stations or even satellites circling the earth. For a system like GPS, position information is extracted from time information. In navigation, positioning accuracy of three meters requires a time accuracy of ten nanoseconds. As a result, systems designers and navigators need to know the characteristics of clocks to be used in their systems in terms of time deviation. Due to noise limitations and physical parameter changes, clocks cannot keep time arbitrarily well. In fact, all clocks show a time deviation with elapsed time. In other words, after a number of clocks are all synchronized with each other and then left alone, each individual clock accumulates a different time reading compared to the other clocks. This deviation of the individual clock readings generally increases with time.

The time deviation ultimately leads the systems engineer to require resynchronization of the clocks. The time needed between resynchronization to achieve a certain time accuracy that meets the systems requirements depends, of course, on the quality of the clock. The main driving force behind having clocks of extreme performance, (excellent time keeping ability) is a desire to have as large a resynchronization interval as possible. This requirement for large resynchronization intervals and excellent clock performance has to do with system needs for reliability for cost reduction, and for subsystem autonomy, in case of complete or partial communications breakdown in the system.

In this paper we shall first define the terms needed to describe clocks, then analyze the effect of time deviation and its proper specification in relation to the clocks, describe the confidence of such a characterization, continue to describe the use of clock models to predict time deviation for times longer than practically available from calibration or measurement intervals and, finally, give a summary of state-of-the-art clocks characterized in their time prediction error.

PERIOD OF AN OSCILLATOR

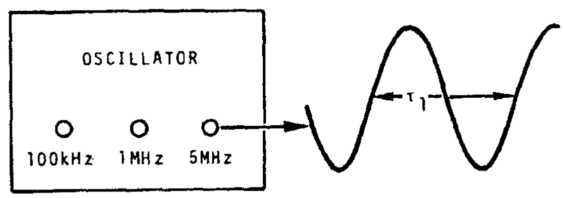


Fig. 1

Describing and Measuring the Frequency Stability of Clocks

The typical clock functions as a precision oscillator with a very stable sinusoidal voltage output with a frequency ν_1 and a period of oscillation τ_1 which is the reciprocal of the frequency, $\nu_1 = 1/\tau_1$ as illustrated in Fig. 1. The goal is to measure the frequency and/or the frequency stability of the sinusoid. The voltage out of the oscillator may be modeled by Eq. (1).

$$V_1 = V_p \sin(2\pi\nu_1 t) \quad (1)$$

Of course, one sees that the period of this oscillation is the number of seconds per cycle or the inverse of the frequency in cycles per second. Naturally, fluctuations in frequency correspond to fluctuations in the period. Almost all frequency measurements, with very few exceptions, are measurements of phase or of period fluctuations in an oscillator, even though the frequency may be the readout. As an example, most frequency counters sense the zero (or near zero) crossings of the sinusoidal voltage, which is the point at which the voltage is most sensitive to phase fluctuations.

One must also realize that most frequency measurements involve two oscillators. In some instances, the oscillator is in the counter. Sometimes one oscillator may be sufficiently better than the other that the fluctuations measured may be considered essentially those of the latter. However, in general, because frequency measurements are almost always dual, it is useful to define:

$$y_1(t) = \frac{\nu_1(t) - \nu_0}{\nu_0} \quad (2)$$

as the fractional frequency deviation of one oscillator, $\nu_1(t)$, with respect to a reference oscillator ν_0, divided by the reference frequency ν_0. Note, that $y_1(t)$ is a dimensionless quantity.

From y(t) we obtain the time deviation, x(t), of an oscillator over a period of time, t, by:

$$x(t) = \int_0^t y(t') \, dt' \quad (3)$$

One may also note that $x(t) = \phi(t)/2\pi\nu_0$, where $\phi(t)$ is the phase deviation of the oscillator under test with respect to the reference oscillator. Since it is impossible to measure instantaneous frequency, any frequency or fractional frequency measurement always involves some sample time, some time window through which the oscillators are observed; a picosecond, a second, or a day may be used, but there is always some sample time. When determining a fractional frequency, y(t), in fact what is being measured is the time fluctuation between some time, t, and a later time, $t + \tau$. The difference in these two time readings divided by τ gives the average fractional frequency over that period:

$$y(t,\tau) = \frac{x(t+\tau) - x(t)}{\tau}. \quad (4)$$

Tau, τ, may be called the sample time or averaging time; e.g., it may be determined by the gate time of a counter.

In many cases, one samples a number of cycles of an oscillation during the gate time of a counter; after the gate time has elapsed, the counter latches the value of the number of cycles so that it can be read out, printed or stored. This results in a delay time before the counter arms and starts again on the next cycle of the oscillation. During the delay time, information is lost. We call this delay "dead time," and in certain instances the measurement result depends on the dead time. However, if the sample time is long compared to the dead time it makes little difference in the result for most cases. [1].

In reality, of course, the sinusoidal output of an oscillator is not pure; but it contains noise fluctuations as well. Frequency stability is defined by the sample variance:

$$\langle \sigma_y^2(N,T,\tau,f_h) \rangle = \left\langle \frac{1}{N-1} \left[\sum_{i=1}^{N} (y_i)^2 - \frac{1}{N} \left(\sum_{i=1}^{N} y_i \right)^2 \right] \right\rangle \quad (5)$$

where y_i denotes the i^{th} fractional frequency measurement of duration τ, $\langle\ \rangle$ denotes an infinite time average, N is the number of frequency readings used in the sample variance, with repetition interval T, and f_h is the bandwidth of the measurement system. Some noise processes contain increasing fractions of the total noise power at lower Fourier frequencies; e.g., for flicker of frequency noise the above variance approaches infinity as $N \to \infty$. This together with the practical difficulty to obtain experimentally large values of N, led to the convention of always using a particular value of N. In recent years, frequency stability has become almost universally understood as meaning the square root of the two-sample variance, i.e., defined as in Eq. (5) for $N=2$, $T=\tau$:

$$\sigma_y^2(\tau) \equiv \left\langle \frac{(y_{i+1} - y_i)^2}{2} \right\rangle \quad (6)$$

For a finite, discrete data set $\sigma_y(\tau)$ can be well approximated by:

$$\sigma_y(\tau) \approx \left[\frac{1}{2(M-1)} \sum_{i=1}^{M-1} (y_{i+1} - y_i)^2 \right]^{1/2} \quad (7)$$

where M is the total number of measurements, each of duration τ; $\sigma_y(\tau)$ is convergent for all noise processes commonly found in oscillators. It should be noted that even for Eq.(6) f_h may be an important parameter and should always be noted. Fig. 2 depicts three different measurement systems which may be used to determine $\sigma_y(\tau)$.

Very often it is useful to resort to a technique illustrated in Fig. 3. The frequencies of the oscillator under test and the reference oscillator are both multiplied to higher frequencies, e.g., into the GHz region, before the beat frequency is obtained and then analyzed. This has the effect of an expansion of the time deviation by the frequency multiplication factor. This relaxes considerably the demand on

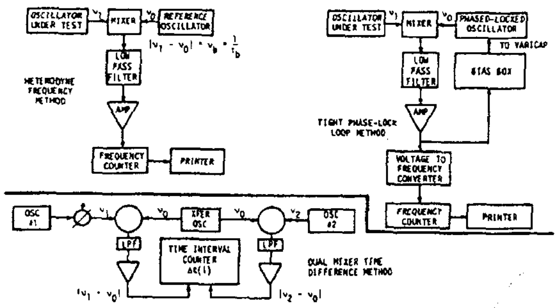


Fig. 2

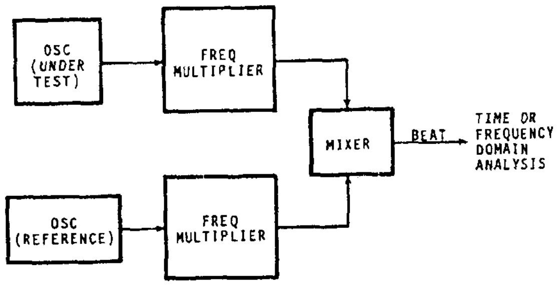


Fig. 3

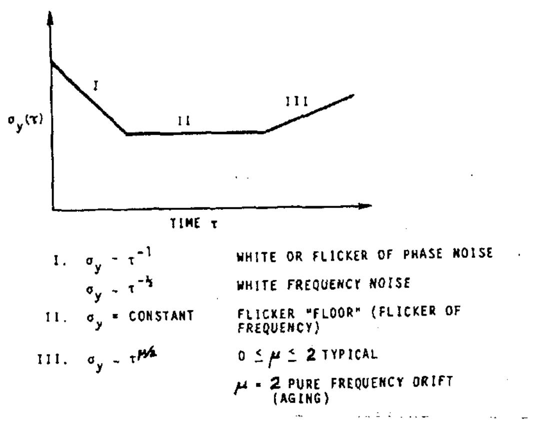


Fig. 4

the beat-frequency analysis equipment but introduces the multipliers as additional noise sources. This frequency multiplication technique is useful for measurements in the frequency domain as well as in the time domain. Commercial equipment based on this technique is on the market.

Measurement and interpretation of clock performance are most commonly done using $\sigma_y(\tau)$. A double logarithmic plot typical for most precise time and frequency standards is shown in Fig. 4. The first part (I) of this plot can be characterized by a reduction in the frequency fluctuation $\sigma_y(\tau)$, with increasing averaging time. This characteristic is determined by the noise processes present in the standard. The second part (II) is called the flicker of frequency floor, which depicts an independence of $\sigma_y(\tau)$ from the averaging time. This flicker floor level depends on the particular frequency standards and is not fully understood in its physical basis, but relates to fluctuations in the value of critical, frequency determining parameters of the standards. The last section (III) of the curve is characterized by a deterioration of $\sigma_y(\tau)$ with increasing averaging time. The precise function of (III) can usually not be determined very accurately because of the long measurement times needed in order to obtain statistical confidence. Also, the clock performance in this region is often not stable during the life of the standard and is subject to environmental influences.

For our purposes it is most important to note that $\sigma_y(\tau)$ is a statistical parameter describing the frequency fluctuations of clocks and oscillators. Being a statistical parameter, its principle usefulness lies in the area of large amounts of data, and it becomes increasingly less useful in cases of insufficient data. For example, if the total available measurement time in an experiment to characterize clock performance is one day, a determination of $\sigma_y(\tau = 1 \text{ day})$ is an obvious impossibility. In fact, there will be no adequate statistical confidence down to sampling times of τ = 0.1 day (compare the more extensive coverage of this subject later). Equally important to the problem of insufficient data on the value of the use of $\sigma_y(\tau)$ is the presence of systematic effects on frequency or time deviation. It is generally an inefficient use of data, if a systematic effect is described and displayed using a statistical variable such as $\sigma_y(\tau)$. Often, the most important systematic effect which should be subtracted from the data before the data are analyzed in a statistical sense is linear frequency drift D (see below).

General Model of Time or Phase Deviation

For most precise clocks and oscillators one may write as a useful description of the time deviation:

$$x(t) = x_0 + y_0\, t + \frac{1}{2}\, D\, t^2 + \varepsilon(t) \tag{8}$$

where the first three coefficients are model parameters: t=0 signifies the instant of synchronization, x_0 is an estimate of the synchronization error at t=0, y_0 is an estimate of the fractional frequency offset error at t=0 and D is the estimate of the fractional frequency drift $\varepsilon(t)$ is taken as the residual (usually random) fluctuation around the quadratic model given by the 1st three terms. If $\varepsilon(t) \equiv 0$, and the first three coefficients were know exactly, then one could predict the time deviation, x(t),exactly;i.e.,the time deviation error would be zero. In practice, for a non-ideal model, there will be uncertainties associated with each coefficient which will directly impact ones ability to predict x(t).

The coefficients y_0 and D may usually best be determined by fitting a linear least squares fit to the frequency (derivative of Eq. (8)).

$$y(t) = y_0 + D\,t + \dot{\varepsilon}(t) \tag{9}$$

or for discrete data (i=1,2,3,...M)

$$y_i = y_0 + D\, i\tau + \frac{\Delta\varepsilon(i)}{\tau} \tag{10}$$

For a large number M of data points, the confidence on determining the first two coefficients y_0 and D for classical statistics (white frequency noise) is:

$$C_{y_0} = \frac{2\,s}{\sqrt{M}} \quad \text{and} \quad C_D = \frac{\sqrt{3}\; C_{y_0}}{M\tau}$$

where s is the standard deviation of the y_i around the linear fit. For non-classical statistics, e.g., for both flicker and random walk frequency noise,

$$C_{y_0} = \frac{2}{\sqrt{10}}\,\sigma_y(0.1T) \quad \text{and} \quad C_D = \frac{\sqrt{3}\; C_{y_0}}{M\tau} \tag{11}$$

(See Appendix A for further details). Clearly, one can generally decrease the confidence intervals by increasing the data length, $T = M\tau$.

In what follows we assume that optimum or near optimum time prediction algorithms are being employed by the user. As the topic of optimum prediction has been treated elsewhere we give only a cursory discussion of it in Appendix B along with some useful references for the interested reader. Optimum time prediction is a very important and non-trivial problem for state-of-the-art clocks, so we highly encourage the reader to pursue this topic if there is any doubt as to what he is using. [B1-B6].

Optimum prediction is the attempt to minimize the time prediction error $\bar{x}(\tau_p)$

Let us define the root mean square time prediction error of a clock as:

$$\bar{x}(\tau_p) = \left\langle (\hat{x}(t,\tau_p) - x(t))^2 \right\rangle^{\frac{1}{2}} \tag{12}$$

where $\hat{x}(t, \tau_p)$ is the time error predicted for the clock at time t, having predicted from $t-\tau_p$.

It is reasonable to assume that each of the terms in Eq. (8) is independent, in which case, Eq. (12) yields:

$$\bar{x}(\tau_p) \simeq [x_0^2 + C_{y_0}^2\,\tau_p^2 + \frac{1}{4}\,C_D^2\,\tau_p^4 + \left\langle \varepsilon^2(\tau_p) \right\rangle]^{\frac{1}{2}} \tag{13}$$

where $\left\langle \varepsilon^2(\tau_p) \right\rangle$ is the mean square random time prediction error. Now, if the linear coefficients, y_0 and D, have been subtracted from the data, then one easily realizes that $\sigma_y(\tau)$ is only dependent on the noise fluctuation. Furthermore, the random squared time prediction error for optimum prediction is given by:

$$\left\langle \varepsilon^2(\tau_p) \right\rangle = (k\tau_p\,\sigma_y(\tau_p))^2 \tag{14}$$

where k is weakly dependent on the type of noise and on the ratio of the prediction interval τ_p to the total data length $T = M\tau$, and k (see Appendix C) is usually near unity. Combining equations (13) and (14), we have as a general equation for the time prediction error:

$$\bar{x}(\tau_p) \simeq \tau_p[(\frac{x_0}{\tau_p})^2 + C_{y_0}^2 + \frac{1}{4}(C_D\tau_p)^2 + k^2\sigma_y^2(\tau_p)]^{\frac{1}{2}} \tag{15}$$

Combining with equation (11) gives:

$$\bar{x}(\tau_p) \simeq \tau_p[(\frac{x_0}{\tau_p})^2 + 0.4\sigma_y^{\,2}(\tau_L) + 0.3\sigma_y^{\,2}(\tau_L)(\frac{\tau_p}{T})^2 + k^2\sigma_y^{\,2}(\tau_p)]^{\frac{1}{2}} \tag{16}$$

where $\tau_p < \tau_L$ and τ_L is the upper limit of a useful sampling time still giving sufficient confidence in the result. For Eq. (16) and all the following discussion we have chosen $\tau_L = 0.1T$ which is equivalent to a 30% (1σ) confidence interval; i.e.,

$$\sigma_y(\tau_L)_{\text{actual}} = \sigma_y(\tau_L)_{\text{meas}} \quad (1 \pm 0.3) \tag{17}$$

Equation (16) is not critically dependent on the assumption $\tau_p < \tau_L$, and as τ_p approaches T one still obtains good estimates for the prediction error $\bar{x}(\tau_p)$.

<u>Characterization of Clocks from Insufficient Data</u>

A systems designer for navigation systems as well as the clock designer is very often faced by the same predicament: Predicting or assuring the time keeping performance of a clock well beyond the range of available test data. Examples are: a) time deviation is to be predicted out to one day but the system only allows the acquisition of clock data out to $\tau = 10^3$ seconds, or b) a specification may read that a clock should not exceed a time error of $x = 1$ nanosecond at $\tau = 10$ days. It is unrealistic to expect from the test facility or even from the clock designer to have statistically significant data for more than $\tau = 1$ day. We believe that the solution to this predicament is the use of clock models in connection with the actual data. In other words, if insufficient data are available we use, based on the available data, a clock model to put an upper boundary on the prediction error for time periods which may significantly exceed the available data length. Which clock model to use is, of course, an important question. We propose to use a worst likely clock model. For the purpose of further discussion we define a sampling time τ_L which is the sampling time for which $\sigma_y(\tau_L)$ is still available with reasonable confidence. We suggest as before $\tau_L = 0.1T$ (see Eq. (17)). In other words, the $\sigma_y(\tau)$ plot as measured on a particular clock is considered valid with adequate statistical confidence out to $\tau = \tau_L$. For the region where τ_p is larger than τ_L a clock model is used to predict time dispersion. Because of the absence of suitable data beyond τ_L our choice of a suitable clock model is based on a wide variety of experiments for several different kinds of clocks. For $\tau_p > \tau_L$,

the main contribution to the time deviation will usually be the uncertainty in the frequency C_{y_0}, in the drift coefficient C_D, and in the random fluctuations (typical worst case is random walk of frequency; $\sigma_y(\tau) \sim \tau^{1/2}$ around a linear regression on the frequency). One may show (See Appendix D) that the time prediction error due to these contribution is:

$$\bar{x}(\tau_p) \simeq \tau_p\, \sigma_y(\tau_L)\left[0.4+0.3\left(\frac{\tau_p}{T}\right)^2 + 1.5\,\frac{\tau_p}{\tau_L}\right]^{1/2} \quad (18)$$

for $\tau_p > \tau_L$.

Very often we will have the case of a specification at some long time τ_p, and the question of what value for $\sigma_y(\tau)$ is required to fulfill this specification when τ_p is much larger than τ_L or T. For example, we can calculate from Eq. (18) for the GPS-like requirements of $x(\tau_p) = 10$ nanoseconds at $\tau_p = 10^6$ seconds, for $\tau_L = 10^5$ seconds (approximately one day), and $T = 10^6$ s, that $\sigma_y(\tau_L) = 2.5 \times 10^{-15}$.

Now it should be clearly stated that in addition to the above description of causes for time dispersion, one must be aware that environmental parameter changes may directly affect the frequency and time of a clock. These environmental parameters may include temperature, magnetic field, pressure, shock, etc. As each clock may have a different dependence for any one of these parameters from another clock, it is incumbent upon the user to determine these and to properly account for them; i.e., each user's environment may be unique.

Summary of the Performance of State-of-the-art Clocks

In Fig. 5 and Fig. 6 we are depicting the stability performance in terms of $\sigma_y(\tau)$ of currently available state-of-the-art standards. This includes rubidium, quartz crystal, cesium and hydrogen standards. This figure is taken from reference [2]. In Fig. 7 and Fig. 8 we are depicting respectively the time prediction error $\bar{x}(\tau_p)$ of these standards taking τ_L as the value of τ at the right end of the solid line for the $\sigma_y(\tau)$ plot for each clock.

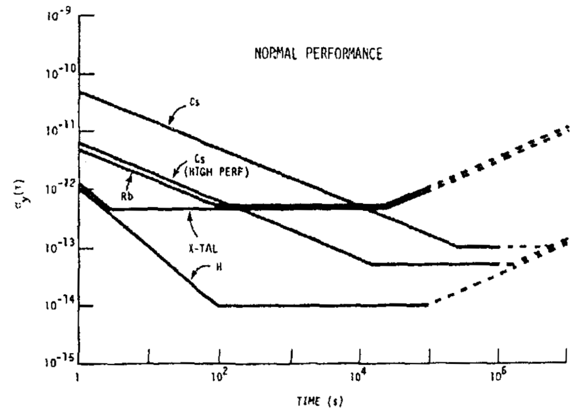


Fig. 5

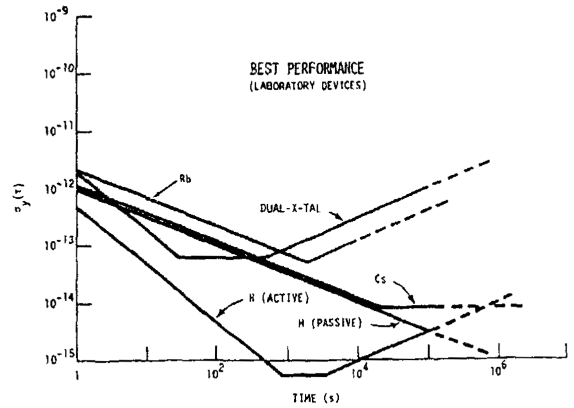
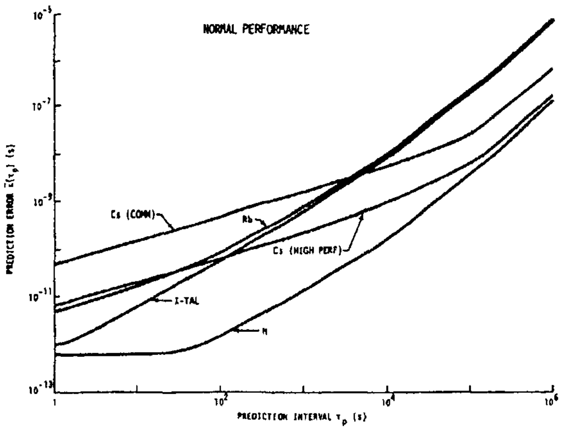


Fig. 6



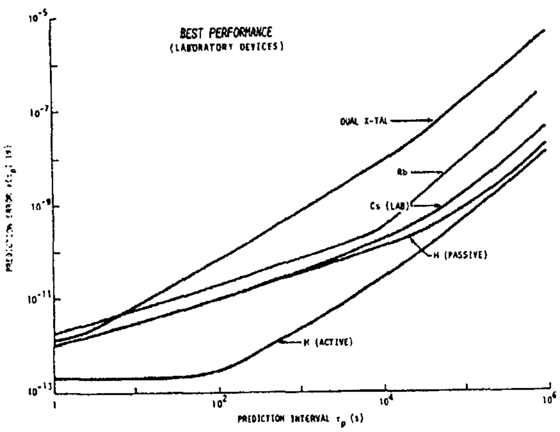


Fig. 5. Frequency stability of commercially available frequency standards (clocks) Cs ≡ cesium beam Rb ≡ rubidium gas cell, x-tal ≡ quartz crystal oscillator, and H ≡ hydrogen maser (active); H is available by special acquisition.

Fig. 6. Some of the best frequency stability performance for laboratory constructed frequency standards: Dual-X-tal ≡ quartz crystal oscillator frequency locked to a special quartz crystal resonator, Cs ≡ primary cesium beam frequency standard, H ≡ hydrogen maser, Rb ≡ rubidium (lab).

Fig. 7 and 8. The rms prediction error for the data shown in Figs. 5 and 6, respectively using Eq. (16) and (18) and assuming a data length $T = 10\tau_L$, where τ_L is the maximum τ values in Figs. (5) and (6) (Solid line) for which statistically significant data are available.

Summary

As precision clocks are being used more and more for accurate navigation (determining position and velocity), an important need has arisen for having a measure of the time deviation $x(t)$, of these clocks, since time and distance are directly related. This time deviation can be predicted only within some uncertainty, the time prediction error $\tilde{x}(\tau_p)$. Since both of these measures, the time deviation and the time prediction error, are extremely important to accurate navigation as well as to other important fields, we have developed these measures based on reasonable models of clock behavior. Furthermore we have constrained these measures to be dependent only on commonly available measurement data.

Specifically, one may write for the model of the time deviation of a clock

$$x(t) = x_0 + y_0\, t + \frac{1}{2} D\, t^2 + \varepsilon(t). \qquad (19)$$

The coefficients for the first three terms are based on the systematic characteristics of a clock and its measurement system: an estimate of the synchronization error, the normalized frequency offset, and the normalized frequency drift. The synchronization error -- though very important -- almost always has its major contribution from the measurement system and may be assumed zero intrinsic to a clock as was done in Fig. (7) and (8). The last term represents the usually random time fluctuations around the quadratic and will significantly influence one's ability to determine the systematic coefficients as will also the total data length, T. The random frequency fluctuations are well characterized by a $\sigma_y(\tau)$ versus τ analysis.

The root mean square uncertainty (the prediction error $\tilde{x}(\tau_p)$) with which one can predict the time deviation $x(t)$, from a time $t-\tau_p$ is given by

$$\tilde{x}(\tau_p) \approx \tau_p\left[\left(\frac{x_0}{\tau_p}\right)^2 + \sigma_y^2(\tau_L)\left(0.4 + 0.3\left(\frac{\tau_p}{T}\right)^2\right) + k^2\sigma_y^2(\tau_p)\right]^{\frac{1}{2}} \qquad (20)$$

for $\tau_p < \tau_L$,

where we have chosen $\tau_L = 0.1T$, and k (see Appendix C) is a coefficient near unity. The initial synchronization error, x_0, though not intrinsic to the clock, can be extremely important in any application. It is incumbent on the user to estimate and include this error from his measurement system, etc. For example, the curves shown in Fig. (7) and (8), would be significantly different by just $x_0 = 1$ ns.

Proper modeling allows one to estimate error even with limited data ($\tau_p > \tau_L$). In this case, Eq. (20) is still valid, however, the last term changes into

$$k^2\sigma_y^2(\tau_p) \xrightarrow[\tau_p > \tau_L]{} 1.5\, \sigma_y^2(\tau_L)\, \frac{\tau_p}{\tau_L} \qquad (21)$$

The choice of $\tau_L = 0.1T$ implies a 30% (1σ) confidence interval, if the above eqations are used. Using Eq. (20) and (21) thus allows one to estimate the predictive error of a clock for essentially all values of interest to the design engineer.

Acknowledgements

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Appendix A: Confidence on the Coefficients for a Linear Regression

Given the equation:

$$y(i) = y_0 + D \cdot i\tau + \frac{\Delta\varepsilon(i)}{\tau} \tag{A1}$$

for i = 1,2,3,...M and assuming white frequency noise, the standard error of the regression coefficients may be shown respectively to be [see refs. A1, A2]:

$$C_{y_0} = s\sqrt{\frac{2(2M+1)}{M\ (M-1)}} \tag{A2}$$

$$= \frac{2s}{\sqrt{M}} \text{ for M large,}$$

and

$$C_D = \frac{\sqrt{12}\ \ s}{\sqrt{M(M+1)(M-1)}} \tag{A3}$$

$$= \frac{2\sqrt{3}\ s}{M\sqrt{M}} = \frac{\sqrt{3}\ C_{y_0}}{M} \text{ for M large,}$$

$$\text{where } s = \left[\frac{1}{M-2}\left(\sum_{i=1}^{M}(y_i)^2 - \frac{1}{M}\left(\sum_{i=1}^{M} y_i\right)^2\right)\right]^{1/2} \tag{A4}$$

Note, s is the standard deviation with 2 degrees of freedom removed.

For non-classical statistics Eq. (A2) must be modified appropriately. To account for cases where either flicker or random walk frequency noise is a good model consider the following: Take as the maximum τ value $\tau_L = 0.1T$ for a reasonable confidence interval (~ 30%) for $\sigma_y(\tau_L)$.

For white frequency noise,

$\sigma_y(\tau) = s$, $\sigma_y(T) = s/\sqrt{M}$, and

$\sigma_y(\tau_L) = \sqrt{10/M}\ s$, therefore $2\ s/\sqrt{M}$

$= 2\ \sigma_y(\tau_L)\ /\sqrt{10} = C_{y_0}$. In addition to being valid for classical white frequency noise $\sigma_y(\tau_L)$ will be appropriately statistically responsive to flicker and random walk frequency noise--Eq. (11).

Appendix B: Optimum Time Prediction

What is optimum prediction depends on the assumed models for the clocks as well as on what is being minimized. There are several references which deal with this problem [B1, B2, B3, B4, B5, B6].

At this point we would caution the reader that it is possible to over-model the data and end up with a prediction routine that may be near optimum as compared to a simulated process for prediction times much shorter than the data length but may be very bad for times of the order of or longer than the data length. We would suggest using prediction algorithms which are not critically model or data-length dependent, even though they may not be as near optimum as would be given by a simulated model. Real oscillators never ideally fit the models, and it is important for algorithms not to be sensitive to departures from the models.

Appendix C: Dependence of Time Dispersion on Noise Type and Data Length

For a $\sigma_y^2(\tau) \sim \tau^\mu$ diagram, μ typically takes on values as shown in Table C1 for most precision oscillators. The constant k in the text (equations 14, 15, 16, and 20) is slightly μ and T/τ_p dependent, as can be seen from the table; T/τ_p is the number of samples going into the two-sample variance and is assumed here to be equal to 10 or more. If the actual value of k is used in the above equations, then it will account for the upper 1σ confidence limit resulting from the noise type and the data length [see refs. B7, B8].

Table C1

μ	Name	$k \equiv k(\mu, T/\tau_p)$
-2	White or flicker phase noise	$\frac{1}{\sqrt{3}}\left(1+\sqrt{\frac{\tau_p}{T}}\right)$
-1	White frequency noise	$1 + .87\sqrt{\frac{\tau_p}{T}}$
0	Flicker frequency noise	$\frac{1}{\sqrt{\ln 2}}\left(1 + .77\sqrt{\frac{\tau_p}{T}}\right)$
+1	Random Walk freq. noise	$1 + .75\sqrt{\frac{\tau_p}{T}}$

Appendix D: Random Walk Frequency Prediction Errors

For the case where the random frequency fluctuations are characterized by a model around a linear regression which is random walk, the optimum time-prediction error is given by:

$$\bar{x}(\tau_p) = \tau_p \cdot \sigma_y(\tau_p) \qquad \text{(D1)}$$

Now if $\tau_p > \tau_L$ & $\sigma_y(\tau_L)$ is the stability for the longest sample time for which we have reasonable confidence, then

$$\bar{x}(\tau_p) = \frac{\tau_p^{3/2}}{\tau_L^{1/2}}\,\sigma_y(\tau_L) \qquad \text{(D2)}$$

and considering the upper confidence interval

$$\bar{x}(\tau_p) = 1.24\,\tau_p\sqrt{\frac{\tau_p}{\tau_L}}\,\sigma_y(\tau_L). \qquad \text{(D3)}$$

Equation (D3) when squared is the last term in equation (18). The first two terms in equation (18) are the same as in equation (16).

Addendum to paper "Time Deviation and Time Prediction Error for Clock Specification, Characterization, and Application," authored by David W. Allan and Helmut Hellwig [1]. (Revised 5 March 1981)

David W. Allan

The above cited paper develops a method of estimating a clock's rms time error over a prediction interval, τ_p [1]. The model assumed for the time departure of the clock is:

$$x(t) = x(o) + y(o)\cdot t + \frac{1}{2} D\cdot t^2 + \varepsilon(t), \tag{1}$$

where x(o) and y(o) are the synchronization and syntonization errors, respectively, at t = 0, D is the modelable frequency drift, and $\varepsilon(t)$ is the random residual time fluctuations for the clock. Given a data set of length T for the clock, a $\sigma_y(\tau)$ vs. τ analysis can be performed to characterize the levels and kinds of noise processes which model the clock behavior. In the development, the longest sample time τ (denoted τ_L) for which significant statistical confidence is available is $\tau_L = 0.1$ T. It is further assumed that optimum time prediction techniques will be used for estimating time from the clock.

Long-term data from clocks are often hard to come by. Because of this, the authors of the above cited paper have employed reasonable and conservative models which are applicable to essentially all precision clocks for sample times, τ, larger than τ_L. Using the above models and assumptions, the authors developed two equations for the rms time error of prediction $\bar{x}(\tau_p)$--one for $\tau_p < \tau_L$ and one for $\tau_p >_L$. The above approach and insights provided an opportunity for a vendor to meet a long-term user rms time error specification without having extensive long-term data as heretofore would have been necessary.

However, equations (16) and (18) in the above cited paper give different values for $\bar{x}(\tau_p)$ in the vicinity of $\tau_p \cong \tau_L$. This is true because the last term in equation (18) is the asymptotic limit for $\tau_p >> \tau_L$. It is useful, therefore, to combine the two equations into one general equation, which retains the proper asymptotic limit and is statistically valid for all τ_p of the order of the data length, T, and shorter:

$$\bar{x}(\tau_p) = \tau_p \left\{ \frac{a^2}{3\tau_p^2} + \frac{b^2}{\tau_p} + 1.4c^2 + \sigma_y^2(\tau_L) \left[0.4 + 1.5 \left(\frac{\tau_p}{\tau_L}\right)^{\mu} + 0.003 \left(\frac{\tau_p}{\tau_L}\right)^2 \right] \right\}^{\frac{1}{2}}, \qquad (2)$$

with $\mu = +1$ for $\tau_p < \tau_L$ and where a, b, and c are the values of $\sigma_y(\tau = 1s)$ for each of the independent noise processes: white or flicker phase noise ($\sim\tau^{-1}$), white frequency noise ($\sim\tau^{-\frac{1}{2}}$), and flicker frequency noise ($\sim\tau^0$), respectively. The value of μ is usually taken to be 1 as is argued in the cited paper, i.e., conservative model (random walk of frequency). In some cases, it may be argued that the value for μ should be different from +1 for $\tau_p \geq \tau_L$ as will be explained below.

Hence, if $\mu = 1$, all that is needed is the estimates of the noise levels from a $\sigma_y(\tau)$ diagram in order to calculate an estimated rms time error of prediction for any $\tau_p \lesssim T$. Taking $\sigma_y^2(\tau)$ as being chi-squared distributed and the actual rms error from the clock as being normally distributed, the confidence that the error will actually be less than calculated by $\bar{x}(\tau_p << \tau_L)$ is about 68% (~1σ) for $\tau_p << \tau_L$. For $\tau_p \sim \tau_L$, the coefficient 1.5 for the middle term in square bracket of the equation gives a confidence of the estimate for $\sigma_y^2(\tau_L)$ of 90%, 86.5%, and 82% for white frequency noise, flicker frequency noise, and random walk frequency noise, respectively. In the range of $\tau_p \sim T$, it is difficult to quantify the probabilities, but be it sufficient to say that because of the conservative model assumption of random walk frequency for $\tau_p > \tau_L$ (μ = +1), the actual clock errors should be well within 1σ (68%) of a normal distribution [2].

Though the above equation is conservative and statistically sound for $\tau_p \lesssim T$, it should be carefully noted that it does not address questions of clock lifetime, of failure mechanisms, or of some systematic deviations such as strong periodic fluctuations, and abnormal environmental influences. These effects along with asynchronization effects and measurement system purturbations must be considered separately for each clock.

In general, combining the above, the $\bar{x}(\tau_p)$ equation -- assuming normal clock behavior over the prediction interval τ_p of interest -- will give conservative (at least 1σ) estimates of time prediction errors for all τ_p up to and of the order of T. Even when τ_p is much larger than T, reasonable estimates may be obtained provided that there is a logical reason for extrapolating $\sigma_y(\tau)$. Even though statistical confidence is lost as τ_p gets much larger than the data length, T, equation (2) is designed to still give conservative upper limit (1σ) estimates of a clock's performance capability with $\mu = 1$.

It is worth noting that there is a way of estimating μ from the data. This may be useful in the case of a long data run where obtaining data is very costly, it may be well worth while considering the following.

The proper value of μ is determined from the dependence of $\sigma_y^2(\tau)$ on τ for $\tau \geq \tau_L$; i.e., $\sigma_y^2(\tau \geq \tau_L) \sim \tau^\mu$. Although one cannot with confidence obtain this directly from the data of length $T = 10\tau_L$, there is a good statistical estimation method conveniently available; i.e., the bias function $B_1(N, \mu) = \langle\sigma^2(N, \tau)\rangle / \sigma_y^2(\tau)$, where the numerator is the expectation value of the squared standard deviation taken over N values [3], and is given by:

$$B_1(N, \mu) = \frac{N(N^{\mu}-1)}{2(N-1)(2^{\mu}-1)} \tag{3}$$

In our current case if $\tau = \tau_L$, then N = 10 and μ may be simply approximated by the following:

$$\mu \cong \frac{B_1(N=10,\mu)}{5} \tag{4}$$

In other words one can determine the standard deviation of the 10 adjacent frequency values, each of them sampled for τ_L, square it and divide by 5 $\sigma_y^2(\tau_L)$ for a simplified estimate of μ (exact for μ = 1). Table 1 lists some useful values for the transcendental equation (3):

Table 1

$B_1(N=10,\mu)$	18.3	13.9	10.6	8.2	6.4	5.0	4.0	3.2	2.6	2.2	1.8	1.6	1.4	1.2	1.1	1.0
μ	2	1.8	1.6	1.4	1.2	1.0	0.8	0.6	0.4	0.2	0.0	-0.2	-0.4	-0.6	-0.8	-1.0

flicker walk frequency (μ = 2); random walk frequency (μ = 1.0); flicker frequency (μ = 0.0); white frequency (μ = -1.0)

The approximation given in (4), though exact for μ = 1, gives conservative results for μ greater than or less than μ = 1. $B_1(N,\mu)$ can take on values less than 1 for certain noise processes, but those are not of interest in this context as the term associated with the uncertainty in the frequency drift will predominate. It is recommended that if $B_1(N=10,\mu) \leq 1.8$ (flicker frequency noise), then the value of μ should be 0, because of the ubiquitous nature of flicker noise and the lack of statistical information to confirm performance better than that. It may be shown that $B_1(N,\mu)$ is a good measure of the low Fourier frequency components in the data [3,4].

Equation (2) has been developed as a tractable method of arriving at conservative estimates of the rms time error of prediction for a large range of values of the prediction time τ_p. The equation is designed so that the input parameters are easily obtainable. It should thus find usefulness for not only systems engineers, but also for the users and the manufacturers, and hopefully will provide a communication bases between these groups.

I wish to thank Dr. James A. Barnes, Dr. Samuel R. Stein, and Dr. Fred L. Walls for very stimulating suggestions and comments to this report. In addition, I wish to thank Mr. Charles Gray of Martin-Marietta who stimulated the above work. During a careful reading of the above cited paper, he found some departures from equations (16) and (18) in figure (7) of that paper for Cs and H.

REFERENCES

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[2] J. A. Barnes, private communication and by permission some of his unpublished work.

[3] "A Coordinate Frequency & Time System," G. E. Hudson, D. W. Allan, J. A. Barnes, R. Glenn Hall, J. D. Lavanceau, and G. M. R. Winkler, Proc. 23rd Annual Symposium on Frequncy Control, Fort Monmouth, NJ, pp. 249-262, May 1969.

[4] "Statistics of Atomic Frequency Standards," D. W. Allan, Proc. IEEE, Vol. 54, No. 2, pp. 221-230, February 1966.

[5] "Statistics of Time and Frequency Data Analysis," D. W. Allan, John H. Shoaf, and Donald Halford, NBS Monograph 140, Chapter 8, May 1974.

Tables (2), (3), and (4) below are the values for the parameters used in equation (2) to generate the results shown in figures (1), (2), and (3) respectively, for the rms time prediction error.

Table 2

(Normal Performance)

Clock	$\sigma_y(\tau_L)$	τ_L	a	b	c	μ
Cs (commercial)	1×10^{-13}	10^6s	0	4.8×10^{-11}	1×10^{-13}	1
Cs (high perf)	4×10^{-14}	10^6s	0	6.3×10^{-12}	4×10^{-14}	1
Rb	3×10^{-12}	10^5s	0	4.7×10^{-12}	3×10^{-13}	1
H (active)	1×10^{-14}	10^5s	1×10^{-12}	0	1×10^{-14}	1

Table 3

(Best Performance)

Clock	$\sigma_y(\tau_L)$	τ_L	a	b	c	μ
NBS-4	8.1×10^{-15}	4 days	0	2×10^{-12}	6.6×10^{-15}	0
NBS-6	1×10^{-14}	1.5×10^5s	0	1×10^{-12}	1×10^{-14}	1
H (passive)	5×10^{-15}	1.2×10^5s	0	1.7×10^{-12}	0	0
H (active)*	2×10^{-15}	0.7×10^5s	1.7×10^{-13}	3.5×10^{-14}	0	1

*These values are taken directly from a $\sigma_y(\tau)$ plot supplied privately to me by Dr. Robert F. C. Vessot for one of his VLG-11 masers assuming equal contributions of the two measured.

Table 4

(GPS Clocks)

Clock	$\sigma_y(\tau_L)$	τ_L	a	b	c	μ
GPS Spec	2×10^{-13}	10^5s	0	5×10^{-11}	2×10^{-13}	1
GPS Spec	2×10^{-13}	10^5s	0	5×10^{-11}	2×10^{-13}	0
Rb (100 day)	2×10^{-13}	10 days	0	9.7×10^{-12}	1.2×10^{-13}	1
FTS	1×10^{-13}	10 days	0	1.7×10^{-11}	1×10^{-13}	1
H (passive) small	1.2×10^{-14}	4 days	0	1.7×10^{-12}	1.2×10^{-14}	0

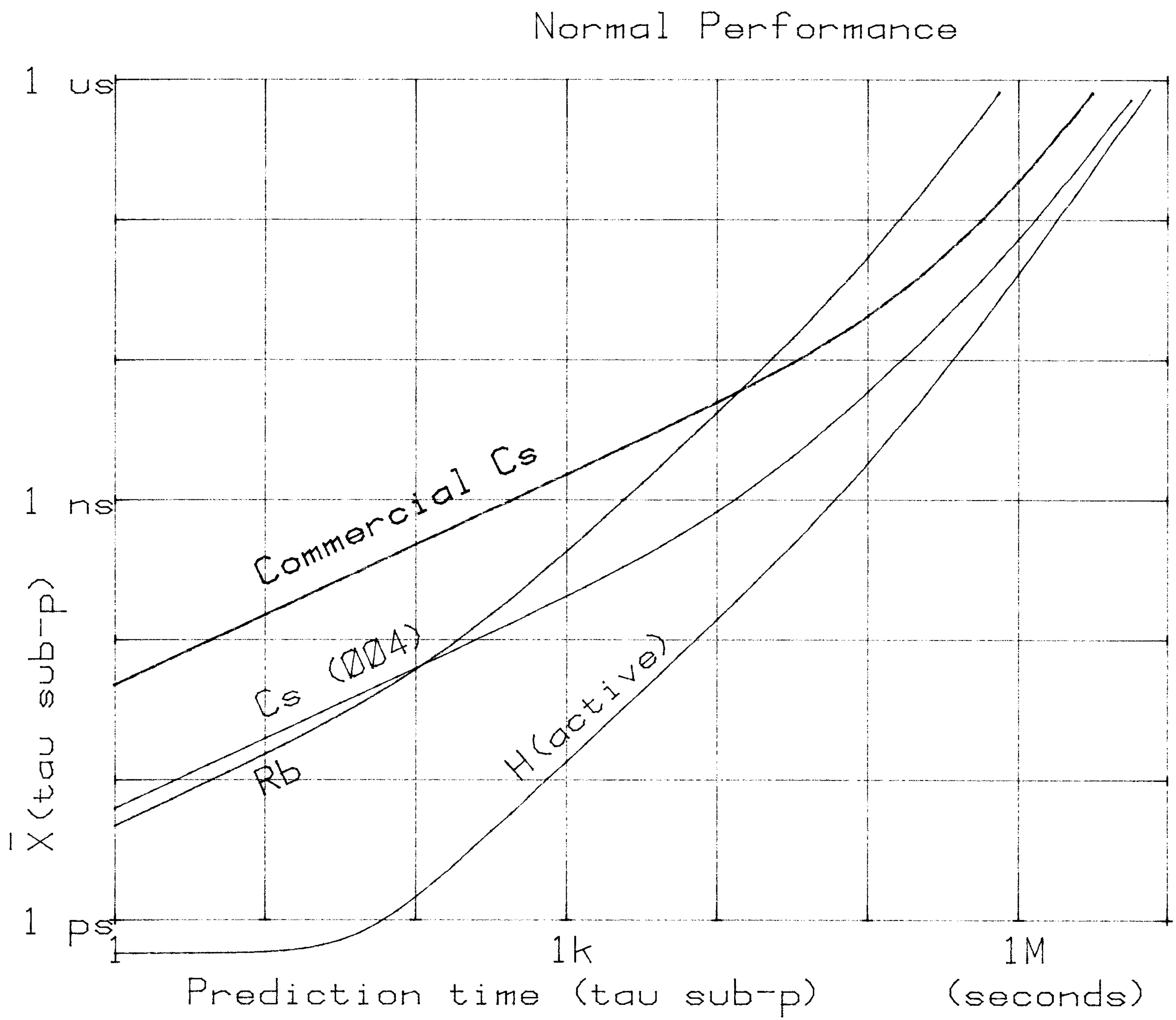
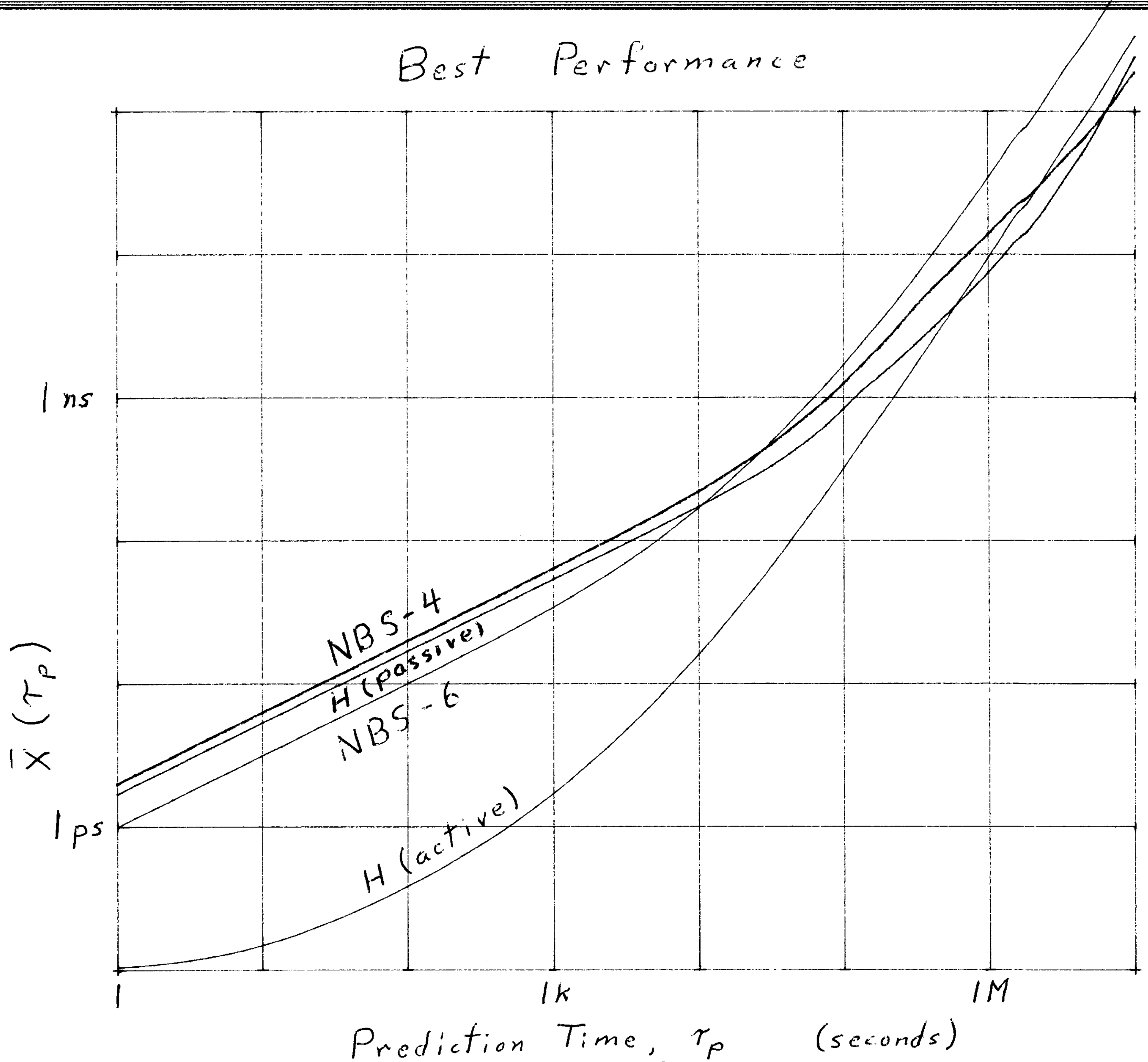

Normal Performance
1 us
1 ns
1 ps
X(tau sub-p)
Commercial Cs
Cs (ØØ4)
Rb
H(active)
1
1k
1M
Prediction time (tau sub-p)
(seconds)

FIGURE 1.



Best Performance
1 ns
1 ps
$\bar{X}(\tau_p)$
NBS-4
H (passive)
NBS-6
H (active)
1
1k
1M
Prediction Time, τ_p (seconds)

FIGURE 2.

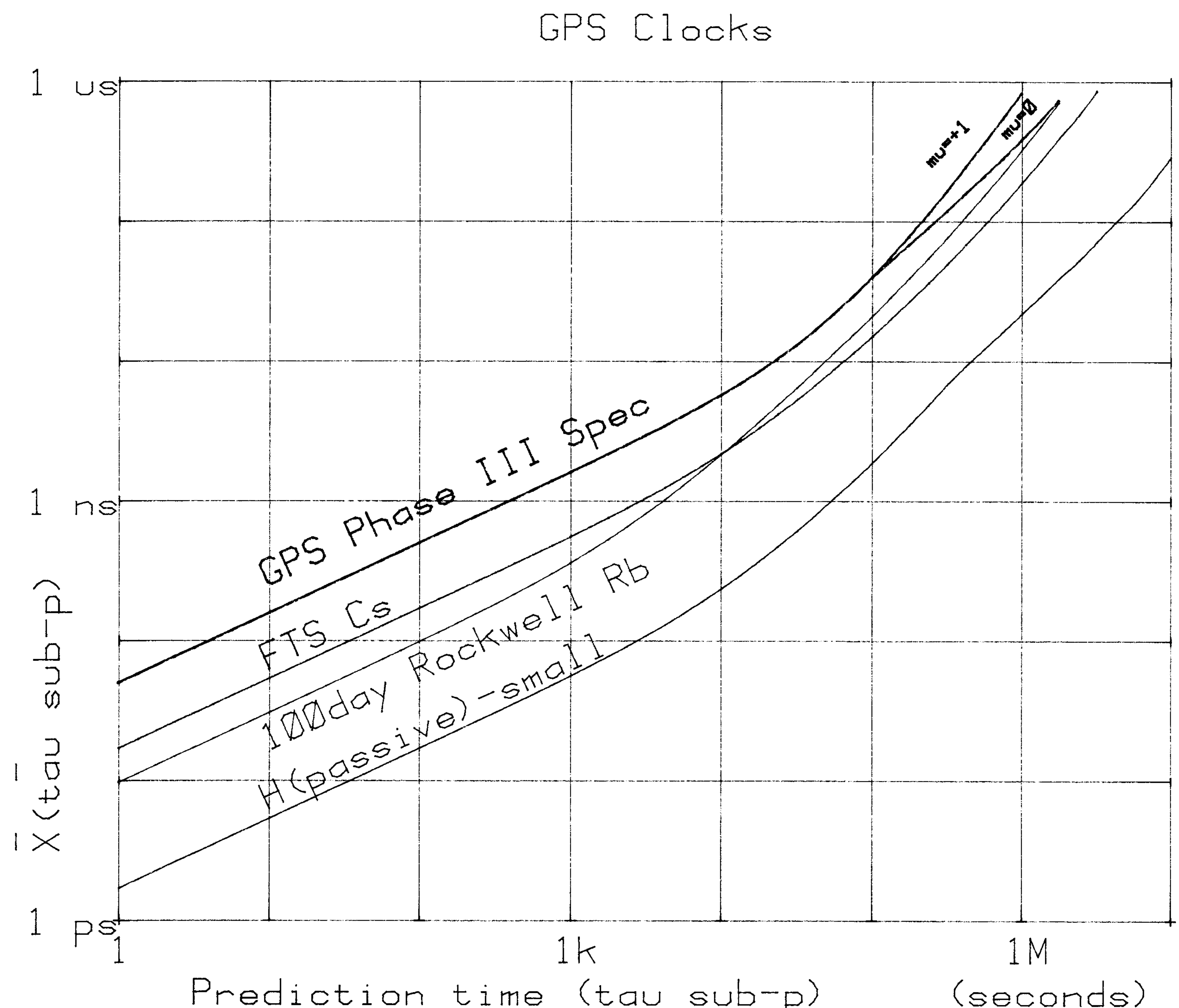

GPS Clocks
1 us
1 ns
1 ps
X(tau sub-p)
GPS Phase III Spec
FTS Cs
100day Rockwell Rb
H(passive)-small
mu=+1
mu=0
1
1k
1M
Prediction time (tau sub-p)
(seconds)

FIGURE 3.