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SIMULATION STUDY USING A NEW TYPE OF SAMPLE VARIANCE

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Abstract

We evaluate with simulated data a new type of sample variance for the characterization of frequency stability. The new statistic (referred to as TOTALVAR and its square root TOTALDEV) is a better predictor of long-term frequency variations than the present sample Allan deviation. The statistical model uses the assumption that a time series of phase or frequency differences is wrapped (periodic) with overall frequency difference removed. We find that the variability at long averaging times is reduced considerably for the five models of power-law noise commonly encountered with frequency standards and oscillators.

1. INTRODUCTION

The most common method of quantifying frequency stability between oscillators is to evaluate the RMS of the fractional frequency changes vs. averaging time τ , dubbed the Allan deviation^[1]. For any sequence of average fractional frequency deviations $\{\bar{y}_i\}$, the widely used quantity $\hat{\sigma}_y(\tau)$ is ideally suited as a reliable, easily interpretable statistic for the characterization of frequency stability for common kinds of FM oscillator noise^[2,3].

There is a considerable literature on various methods and candidate statistics for the characterization of relative oscillator frequency stability. Suffice it to say that for a given system and noise, a statistic can be constructed to be nearly optimum. A single, unified approach will have its compromises. The Allan deviation, however, has a remarkable range of applicability in quantifying frequency and phase stability. This is because as a function of averaging time τ , it is particularly well-suited in identifying the model of the trend in frequency stability or what is called the underlying "power-law" over a range of τ values. The power-law is the slope on a typical log-log $\sigma_y(\tau)$ plot, and $\hat{\sigma}_y(\tau)$ is suitably the RMS prediction error of frequency stability. Predicting the long-term stability of a frequency reference rests ultimately on predicting (correctly identifying) its power-law behavior. For an estimate of stability longer or different than the measurement at hand, simply extrapolate from or directly apply an expected trend (power-law slope). Lastly, we can estimate the evolution of a squared phase error as proportional to τ^2 times the modified Allan variance for

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a uniquely identified power-law slope. This is the time variance or TVAR^[4].

A long-standing problem is that the best statistic, the two-sample Allan deviation, has rather poor confidence at longer and longer τ -values, where confidence is often needed most. A new statistic has been developed which retains the intuitive simplicity of the RMS fractional frequency changes (Allan deviation) and which has improved confidence at long-term averaging times^[5]. The model for the new statistic uses the assumption that a time series of phase or frequency differences is wrapped (periodic) with overall frequency difference removed^[6]. Figure 1 illustrates the procedure. This variance (thus its square root) reduces estimation errors universally seen in previous treatments, thereby providing a better estimate of frequency stability for measurement times longer than say 20% of the data length.

We compare the response of the new statistic (as a variance) to the traditional Allan variance by simulation of the five models of power-law noises commonly encountered with oscillators and frequency standards. Results show that the new variance shows a promise for greatly reduced variability hence uncertainty compared to the traditional Allan variance.

2. DISCUSSION

The sample Allan deviation $\hat{\sigma}_y(\tau)$ and $\text{mod-}\hat{\sigma}_y(\tau)$ are square roots of two types of tau-domain sample variances (AVAR and MVAR)^[1,7]. They are recommended statistics in quantifying frequency stability between oscillators. In certain situations their responses have high variability at long averaging times τ , as indicated by traditional simulation studies using common noise types, because the traditional sample Allan statistics are time-shift dependent. Therefore these statistics have degraded confidence at long averaging times. The method of complex demodulation motivates another statistic which is an improved sample variance for the characterization of frequency stability^[5]. For average fractional frequency fluctuations $\{\bar{y}_{k'}\} = \bar{y}_1, \dots, \bar{y}_{N-1}$ with overall frequency difference removed, this sample variance is given by

$$\hat{\sigma}_{total}^2(\tau) = \frac{1}{N-1} \sum_j^{N-1} \left[\frac{1}{2(M-1)} \sum_{k=1}^{M-1} (\bar{y}_{k+1,j} - \bar{y}_{kj})^2 \right], \quad (1)$$

where $\{\bar{y}_{k',j}\} = \bar{y}_{j+1}, \bar{y}_{j+2}, \dots, \bar{y}_{N-1}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_j$ are spaced by τ_0 and $\{\bar{y}_{k'}\}$ is therefore wrapped and reindexed by j . Series $\{y_{kj}\}$ - with unprimed k - are averages implied over $\tau = m\tau_0$. Hence as with traditional AVAR (and MVAR), the new sample variance $\hat{\sigma}_{total}^2$ is implicitly dependent on dimensionless quantity m , a scale parameter which determines τ and which for efficiency can be limited to rational powers of 2, that is, $2^i = m$, $i=0,1,2,3,\dots$.

Measurements of relative phase differences $\{x_{k'}\}$ are preferred to average frequency $\{\bar{y}_{k'}\}$ as in eq (1). We have $k'=1,2,3, \dots, N$ and separated by interval τ_0 and overall frequency difference removed; therefore $x_1 = x_N$. Furthermore $\{x_{k'}\}$ is wrapped and assumed periodic; hence $x_1 = x_{N+1}$ and we eliminate the increment x_N to x_{N+1} to avoid bias (see figure 1). We have

$$\hat{\sigma}_{total}^2(m\tau_0) = \frac{1}{N-1} \sum_j^{N-1} \left[\frac{1}{2m^2\tau_0^2(M-2m)} \sum_{k'=1}^{M-2m} (-x_{(k'-m)-j} + 2x_{(k')-j} - x_{(k'+m)-j})^2 \right], \quad (2)$$

where the argument in the brackets "[*]" has stride $k'-m$ and is time-shifted by $j\tau_0$ and averaged for all $N-1$ possible shifts. This notation centers the second-difference operation (argument in parenthesis) at k' with a span of $\pm m$ which seems more intuitive especially considering the wrap procedure.

3. STATISTICS COMPARED

The primary reasons for using $\hat{\sigma}_y(\tau)$ are that it is well-known, it is simple to calculate, it is the most efficient estimator for FM noise, and it has a unique value for all τ . The primary disadvantage of using $\hat{\sigma}_y(\tau)$ is that the results can be too conservative, sometimes very optimistic at the long τ -values. It can take much longer than the longest reportable τ -values (often orders of magnitude longer) to accurately quantify the underlying low-frequency variations between the frequency standards being evaluated^[3]. For example, quantifying the frequency stability at, say, τ equals two weeks often requires no less than two months of actual measurement time.

We compare the new sample variance $\hat{\sigma}_{total}^2(\tau)$ (also called TOTALVAR) to traditional AVAR $\hat{\sigma}_y^2(\tau)$ using simulation studies of five common integer power-law noise types. These noise types are white PM, flicker PM, white FM, flicker FM, and random walk FM. A version of $\hat{\sigma}_{total}^2(\tau)$ called $\text{mod-}\hat{\sigma}_{total}^2(\tau)$ exists for MVAR; however since our present emphasis is on confidence at long τ -values, AVAR is of interest. MVAR's advantage is in distinguishing white PM from flicker PM which usually are associated with short τ -values. MVAR has no advantage for flicker PM and beyond, which occur at long τ -values. Furthermore, a chief disadvantage to MVAR is that it only extends to 1/3 the total data length, whereas AVAR extends to 1/2 the same length.

For highly divergent noise types, the new statistic is not expected to be unbiased^[8,9]. However, this report indicates that the new statistic essentially estimates the same unbiased quantity as traditional AVAR for the five common integer power-law noise types but has better confidence than AVAR.

4. GENERATION OF SIMULATED $\{x_{k'}\}$ DATA

Most high level computer program languages can return random variables which we then order as a time series $\{a_n\}$. The usual assumption is that variables are uncorrelated and normally (Gaussian) or uniformly distributed. Thus $\{a_n\}$ forms the basis for a white-noise-of-phase process which is characterized by a constant power spectral density, $S_a(f) \propto f^0$. We build from $\{a_n\}$ the other four noise processes: flicker ($\propto f^{-1}$), random walk ($\propto f^{-2}$), flicker walk ($\propto f^{-3}$), and random run ($\propto f^{-4}$). The treatment of non-integer power law noise types

has recently been explored^[10]. We limit our simulations to the five common integer power laws.

Random walk of phase (RWPM) is equivalent to white noise of frequency (WHFM) and is one integration (single summation) of $\{a_n\}$. Random run of phase (RRPM) is random walk in frequency (RWFM) and is two integrations (double summation) of $\{a_n\}$. These operations are among the simplest autoregressive (AR) procedures.

Flicker processes can be generated using an AR operation but must also include an (integrated) moving average (MA). The ARIMA model used in generating the five integer noise processes is adequately described by

$$x_n = \phi_1 x_{n-1} + \phi_2 x_{n-2} + a_n - \theta a_{n-1}, \quad (3)$$

where a_n is an input random variable and x_n is an output.

For flicker of phase (FLPM):

$$\begin{aligned} \phi_1 &= 1.549, \\ \phi_2 &= -0.56, \\ \theta &= 0.88. \end{aligned}$$

Flicker walk of phase is flicker of frequency (FLFM) and is one integration (single summation) of an FLPM series.

As mentioned, random run of phase (or random walk FM, RWFM) could be adequately realized as only a double summation of $\{a_n\}$ which means $\phi_1=2$ and $\phi_2=-1$, and $\theta=0$ in eq (3). Cleaner representations of RWFM are realized for $\theta=\sqrt{3}-2$ ^[11]. Thus we use:

$$\begin{aligned} \phi_1 &= 2, \\ \phi_2 &= -1, \\ \theta &= \sqrt{3}-2 = -0.268. \end{aligned}$$

For the simulations here, some thought went into initializing each sequence to obtain a representation for the flicker and random run noise types. a_1 was chosen to be between 0 and 1; x_{n-1} , x_{n-2} , and a_{n-1} were derived from the end of previous simulations.

In each of the noise types, the top of figures 2 to 6 show plots of 100 calculations of $\hat{\sigma}_{total}(\tau)$ followed below by plots of 100 calculations of $\hat{\sigma}_y(\tau)$ from the same 100 simulations. At the bottom of each figure is a plot of the square root of the mean of $\hat{\sigma}_y^2(\tau)$ derived from the 100 simulations in order to see its agreement or disagreement with theory, that is, the theoretical square root of a mean of an infinite set. Flicker of phase (FLPM) is the only type which does not have a straight-line (log-log scale) theoretical slope owing to a logarithmic dependence on bandwidth.

5. WHITE PM (WHPM) AND FLICKER PM (FLPM) CASES

For short τ -values, we usually find noise modulation of the phase (not frequency) originating from noisy electronics not involved in the frequency-determining elements. White PM (WHPM) noise is broadband phase noise and has little to do with the resonance mechanism. Stages of amplification are usually responsible for white PM noise. This noise can be kept very low with good amplifier design, hand-selected components, the addition of narrowband filtering at the output, or increasing, if feasible, the power of the primary frequency source.

Flicker PM (FLPM) noise may relate to a physical resonance mechanism in an oscillator, but it usually is added by noisy electronics. This type of noise is common, even in the highest quality oscillators, because in order to bring the signal amplitude up to a usable level, amplifiers are used after the signal source. Flicker PM noise may be introduced in these stages. It may also be introduced in a frequency multiplier or frequency synthesizer.

Figures 2(a) and 3(a) show 100 plots of calculations of the square root of $\hat{\sigma}_{total}^2(\tau)$ for 100 simulations of white PM noise and flicker PM respectively. Equation (2) is used for these calculations and $N=1024$ for each simulation. Each of the simulation averages of two-sample variances at $\tau=1$ is equal to one. Figures 2(b) and 3(b) are traditional square root of maximally overlapped $\hat{\sigma}_y(\tau)$ for the same 100 simulations. The bottom plot is the 100-simulation-total square-root of the mean of the sample Allan variances and shows excellent agreement with theory. The spread in the estimates is greater using AVAR instead of the new statistic $\hat{\sigma}_{total}^2(\tau)$.

White and flicker of PM both exhibit a τ^{-1} slope in $\sigma_y(\tau)$ and hence $\sigma_{total}(\tau)$. These noise types differ from the others in an important regard: their amplitudes are significantly affected by measurement (software and/or hardware) bandwidth^[3]. Because of

this, $\text{mod-}\hat{\sigma}_y^2(\tau)$ or modified Allan variance (MVAR) was invented (for analyzing phase data only, $\{x_k\}$) to take full advantage of the $1/n\tau_0$ slope in the standard variance of $\{x_k\}$ for white PM and $\frac{1}{\ln(n\tau_0)}$ slope for flicker PM. As mentioned earlier, we limit our present

discussion to a comparison between $\hat{\sigma}_y^2(\tau)$ and $\hat{\sigma}_{total}^2(\tau)$. This is because the present interest is an improved confidence at long τ -values where more dispersive noise types are encountered and ultimately limit accurate characterization of frequency stability. Again a significant disadvantage to MVAR is that a single longest reportable τ -value is limited to 1/3 the total measurement time; 50% more time is required for equivalent results using MVAR vs. AVAR. It suffices to say, however, that for white PM and flicker PM, the improvement in confidence in the long term is dramatic using the new statistic $\hat{\sigma}_{total}^2(\tau)$ as shown in figures 2 and 3.

6. WHITE FM (WHFM) CASE

The cases of white FM, flicker FM, and random walk FM are of particular importance since they are physically traceable noise types encountered in virtually all precision frequency standards, and they often occur at long τ -values.

White FM noise ($\sigma_y(\tau) \propto \tau^{-1/2}$) is the type found in common passive-resonator frequency standards. These contain a slave oscillator, often quartz, which is locked to a resonance feature of another device which behaves as a high-Q filter. High quality cesium, rubidium, and passive hydrogen standards have white FM noise characteristics^[12]. Howe has previously presented results using white FM simulation that show that the new statistic TOTALVAR is an improved estimate of the mean-square frequency deviations between oscillators particularly at long τ -values^[5]. Figure 4 reproduces those results for the comparison here.

7. FLICKER FM (FLFM) CASE

Flicker FM ($\sigma_y(\tau) \propto \tau^0$) is a noise whose physical cause is not fully understood but may typically be related to the physical resonance mechanism of an active oscillator, the design or choice of parts used for the electronics, or environmental conditions^[12]. Flicker FM noise is considered the quantum limit of resonance devices^[13]. Flicker FM is common in the highest quality oscillators but may be masked by white FM or even white PM and flicker PM in lower quality oscillators.

Figure 5(a) shows 100 plots of calculations of $\hat{\sigma}_{total}(\tau)$ for 100 simulations of flicker FM noise and figure 5(b) is the same set of calculations using traditional square-root of maximally overlapped AVAR. The square root of the mean of the AVAR's of the 100 simulations as shown in figure 5(c) show a slight downward offset which can commonly occur at $\tau = 512\tau_0 = T/2$. Even though the power law is not exact, it is sufficient for the comparison of the spread in the responses between $\hat{\sigma}_{total}(\tau)$ and traditional $\hat{\sigma}_y(\tau)$. Again, the new statistic is preferred since it is generally less susceptible to large variations at long τ -values.

8. RANDOM WALK FM (RWFM) CASE

Of the five models of power-law noise types, random walk FM noise ($\sigma_y(\tau) \propto \tau^{+1/2}$) is most difficult to measure since its power is concentrated mainly very close to the carrier. This translates to near DC when considering phase differences $\{x_k\}$ or average frequency differences $\{\bar{y}_k\}$. Random walk FM usually relates to an oscillator's physical environment. If random walk FM is a predominant noise type then mechanical shock, vibration, humidity, temperature, or other environmental effects may be causing "random" shifts in the carrier frequency^[14,15].

Figure 6(a) and 6(b) are 100 plots of calculations of the square roots of $\hat{\sigma}_{total}^2(\tau)$ and $\hat{\sigma}_y^2(\tau)$ respectively, for 100 simulations of random walk FM noise. Again,

even though the simulated power-law is assumed to be not exact as interpreted from the square root of the mean of AVAR's in figure 6(c), the important point is the comparison of the spread between square roots of TOTALVAR and AVAR (figure 6(a) and 6(b)). And again, the square root of TOTALVAR is preferred since the spread and skews are reduced at long τ values.

9. CONCLUSION

We compare the response of the traditional sample Allan deviation $\hat{\sigma}_y(\tau)$ with a new similar sample statistic $\hat{\sigma}_{total}(\tau)$ referred to as TOTALDEV (square root of TOTALVAR) for the five models of integer power-law noise types. These integer noise types are white PM, flicker PM, white FM, flicker FM, and random walk FM. Using traditional plots of sigma vs. tau and 100 simulations of each noise type, we find the variability in $\hat{\sigma}_{total}(\tau)$ to be less than $\hat{\sigma}_y(\tau)$ in all cases. As a result, we can expect a reduction in the actual measurement time involved to characterize the long-term frequency stability of a standard or oscillator.

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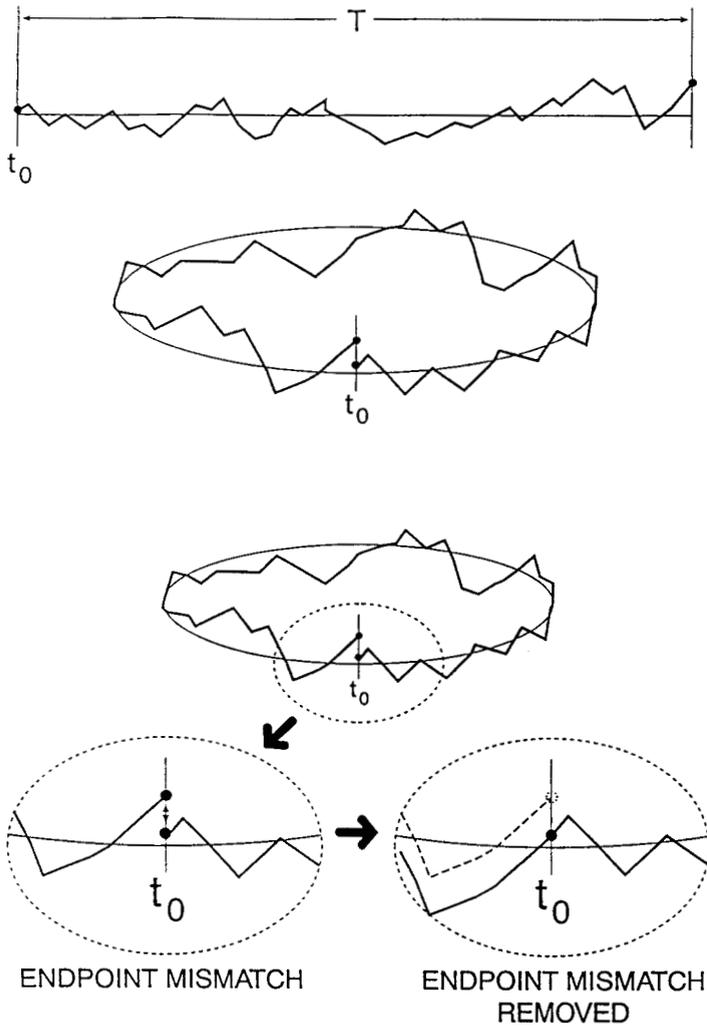


Fig. 1

The new statistic (referred to as TOTALVAR and its square root) uses the model that a time series of phase difference x_t are wrapped with period T and overall frequency difference removed. The periodic assumption means that the data are circularly represented and the time-origin is no longer t_0 but is shiftable by $j\tau_0$ where τ_0 is the minimum measurement interval. TOTALVAR is traditional AVAR averaged over $N-1$ possible shifts. Removal of the overall frequency difference eliminates an end-match step by making $x_1 = x_N$, hence $x_1 = x_{N+1}$ and we eliminate the increment x_N to x_{N+1} to avoid bias. We use

$$\hat{\sigma}_{total}^2(m\tau_0) = \frac{1}{N-1} \sum_j \left[\frac{1}{2m^2\tau_0^2(M-2m)} \sum_{k'=1}^{M-2m} (-x_{(k'-m)-j} + 2x_{(k'-j)} - x_{(k'+m)-j})^2 \right],$$

where the argument in the brackets is traditional AVAR shifted by j .

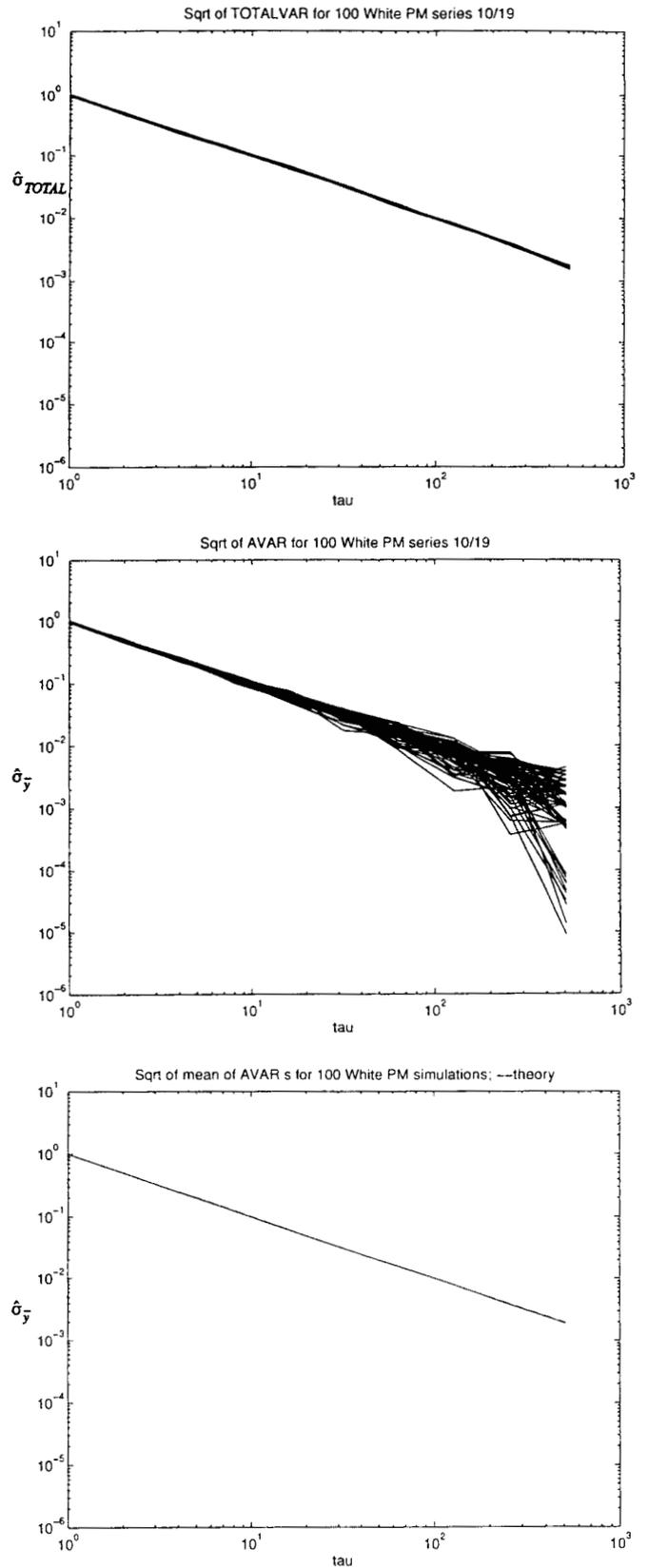


Fig. 2

Top(a): Square root of TOTALVAR calculated for 100 WHPM simulations with unit (two-sample) mean at $\tau=1$.

Middle(b): For comparison, traditional square root of maximally-overlapped AVAR calculated for the same 100 WHPM simulations as used at top for square root of TOTALVAR.

Bottom(c): Square root of 100-total mean of maximally-overlapped AVAR's, an indication of the desired result. Dashed line is the theoretical mean of an infinite set.

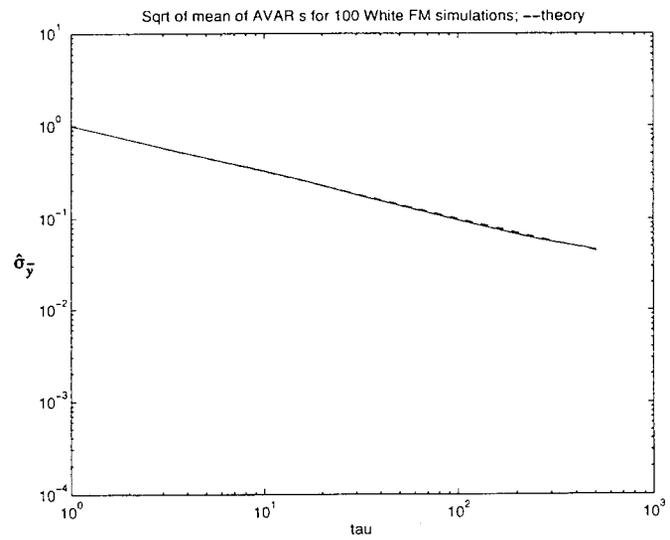
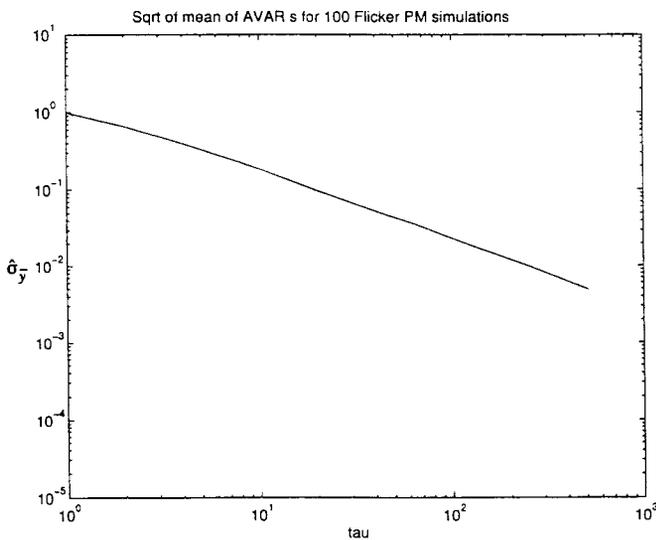
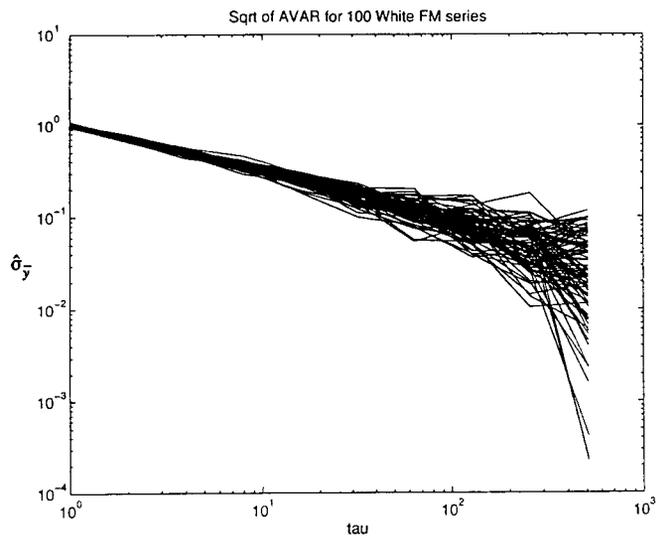
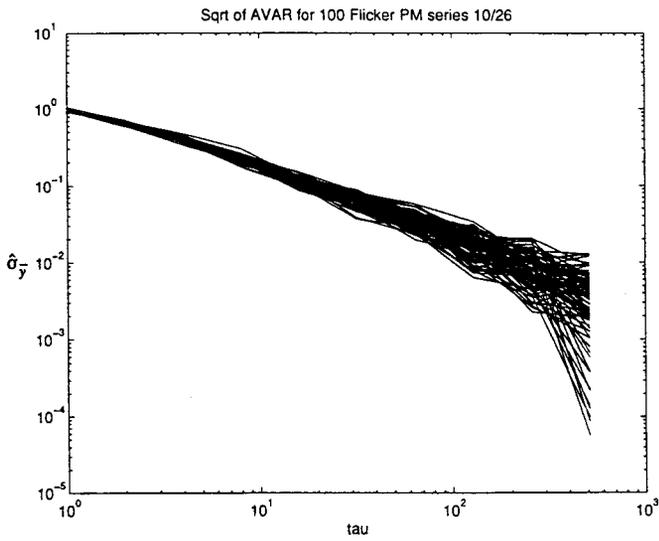
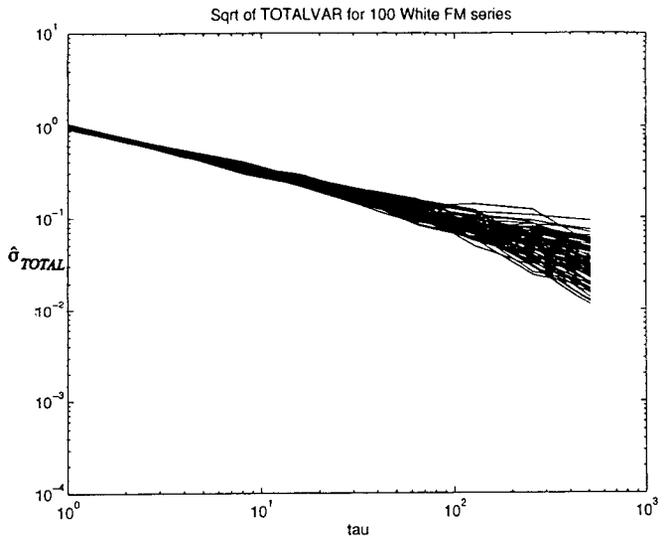
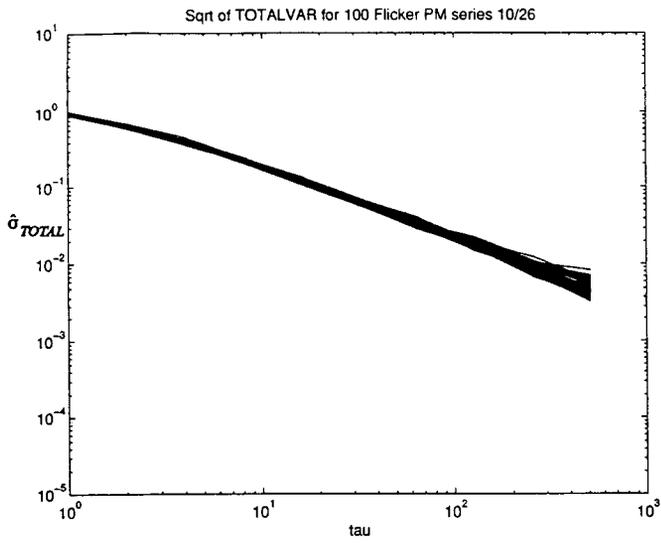


Fig. 3

Fig. 4

Top(a): Square root of TOTALVAR calculated for 100 FLPM simulations with unit (two-sample) mean at $\tau=1$.

Middle(b): For comparison, traditional square root of maximally-overlapped AVAR calculated for the same 100 FLPM simulations as used at top for square root of TOTALVAR.

Bottom(c): Square root of 100-total mean of maximally-overlapped AVAR's, an indication of the desired result.

Top(a): Square root of TOTALVAR calculated for 100 WHFM simulations with unit (two-sample) mean at $\tau=1$.

Middle(b): For comparison, traditional square root of maximally-overlapped AVAR calculated for the same 100 WHFM simulations as used at top for square root of TOTALVAR.

Bottom(c): Square root of 100-total mean of maximally-overlapped AVAR's, an indication of the desired result. Dashed line is the theoretical mean of an infinite set.

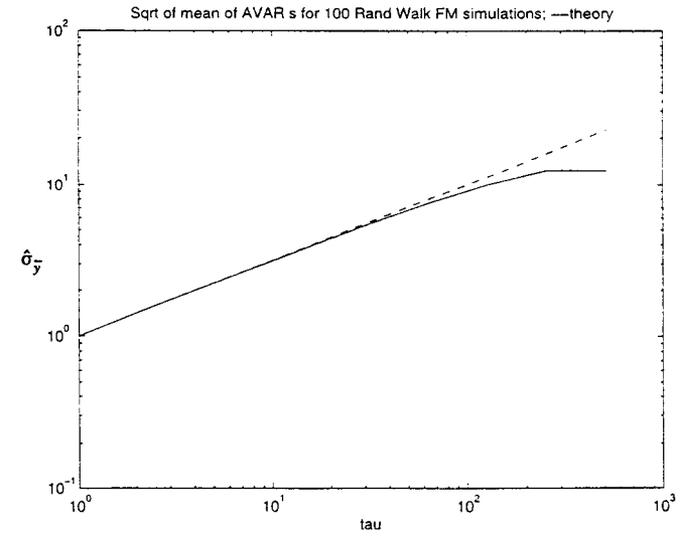
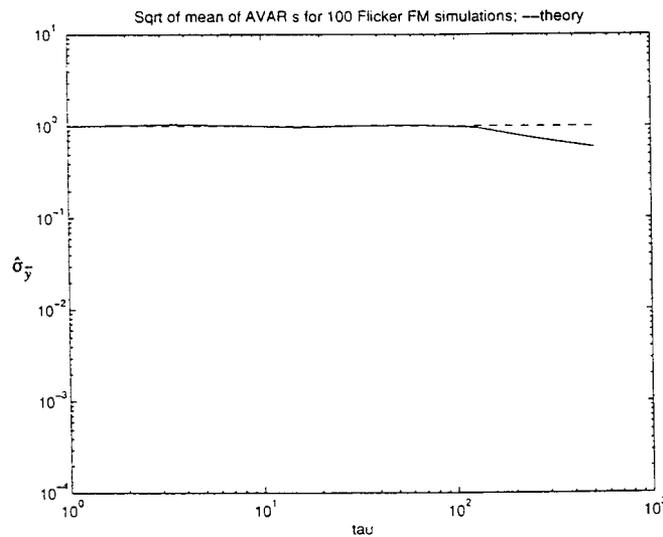
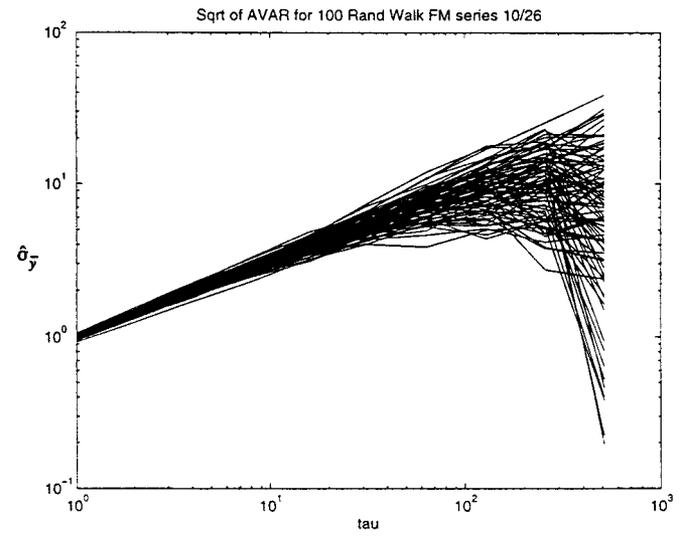
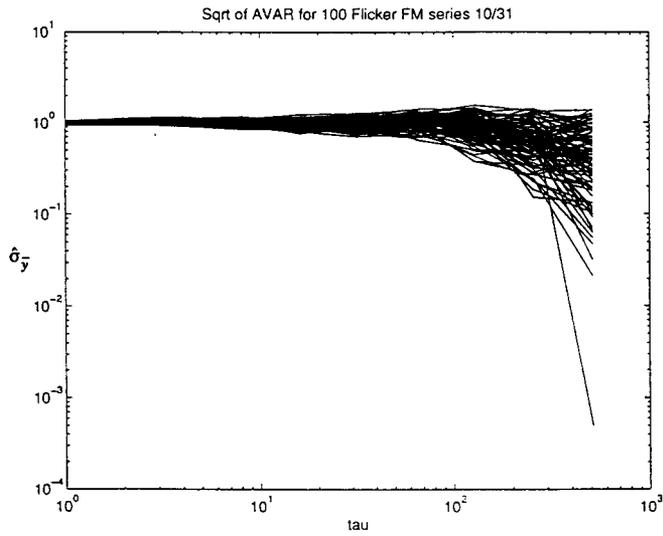
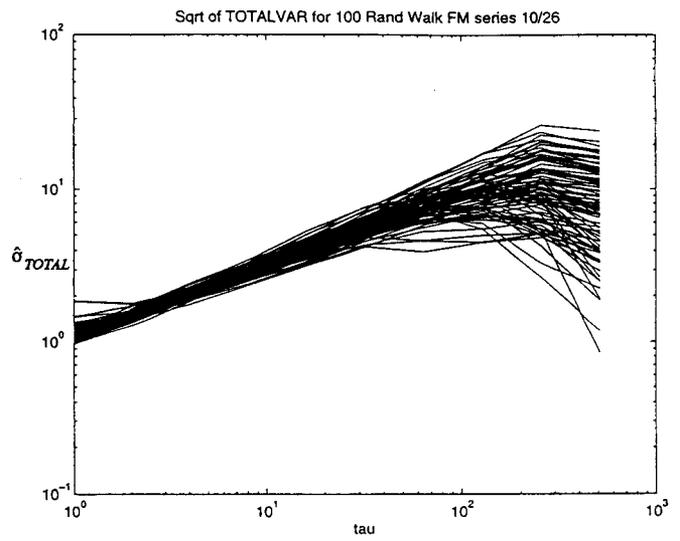
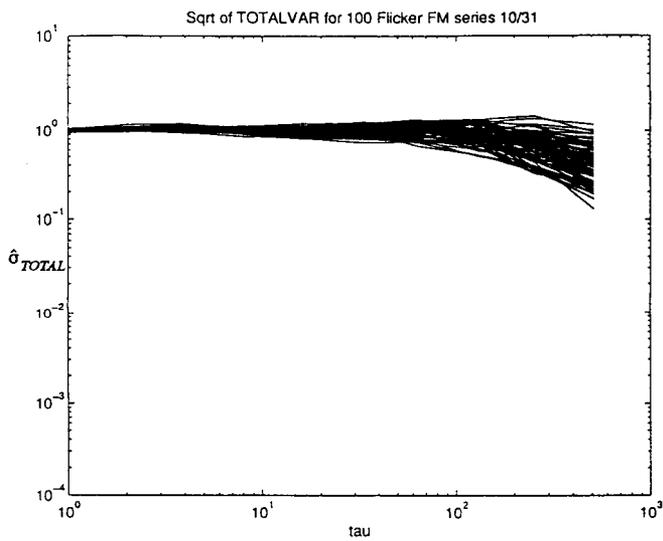


Fig. 5

Fig. 6

Top(a): Square root of TOTALVAR calculated for 100 FLMF simulations with unit (two-sample) mean at $\tau=1$.

Middle(b): For comparison, traditional square root of maximally-overlapped AVAR calculated for the same 100 FLMF simulations as used at top for square root of TOTALVAR.

Bottom(c): Square root of 100-total mean of maximally-overlapped AVAR's, an indication of the desired result. Dashed line is the theoretical mean of an infinite set.

Top(a): Square root of TOTALVAR calculated for 100 RWFM simulations with unit (two-sample) mean at $\tau=1$.

Middle(b): For comparison, traditional square root of maximally-overlapped AVAR calculated for the same 100 RWFM simulations as used at top for square root of TOTALVAR.

Bottom(c): Square root of 100-total mean of maximally-overlapped AVAR's, an indication of the desired result. Dashed line is the theoretical mean of an infinite set.