INFRARED-MICROWAVE FREQUENCY SYNTHESIS DESIGN: SOME RELEVANT CONCEPTUAL NOISE ASPECTS*†

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Abstract

Extremely accurate and precise frequency synthesis into the infrared and visible radiation regions will allow new vistas of metrology. Frequency and time measurements are the basic operations which will be affected, and impact is expected in such diverse areas as length standards and metrology, spectroscopy, timekeeping, communications, and relativistic tests. In addition the set of independent base units of measurement may change, and the speed of light may become a conventional (defined) quantity. The attainment of the desired high accuracy and precision will be easiest and cheapest if there is careful optimization of the synthesis design aspects involving noise. When frequencies in the terahertz region are considered, the linewidth of the signal becomes an important parameter. Due to the low-frequency-divergence of the instability of good signal sources, the concept of the fast linewidth becomes of particular importance. Some of the properties and importance of the fast linewidth in system design are discussed in this paper.

Key Words: Allan variance; Base units; Fast linewidth; Frequency multiplication; Frequency noise; Frequency synthesis; Infrared frequency metrology; Josephson effect; Linewidth; Methane frequency standard; Phase noise; Unified standard

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I. INTRODUCTION

Some short-term goals and some long-term goals for high accuracy and precision in visible-infrared-microwave frequency synthesis are discussed in Section II, together with some of the possible benefits to metrology. A practical engineering model for noises of good signal sources is discussed in Section III, and some implications are discussed for optimum frequency synthesis design as regards noise problems. The dependence of the radio frequency (RF) power spectral density linewidth upon the multiplication factor n is derived and discussed. The significance of the fast linewidth becomes obvious in these treatments. In Section IV, I present some hopes and speculation concerning future developments in frequency synthesis.

II. SOME GOALS AND RECENT ACCOMPLISHMENTS

Joe Wells, David Knight, and Al Risley have just reviewed some of the exciting pioneering work which has been going on in frequency synthesis into the infrared radiation region.[1]

In the talk which follows, I will consider some ideas which have figured in discussions over the past couple of years with David Knight, Bob Kamper, Jim Barnes, Dave Allan, Helmut Hellwig, Al Risley, Don McDonald, John Hall, Len Cutler, and others.

A long-range goal is the creation of simple, reliable, inexpensive means of synthesizing infrared (IR) and visible radiation (VR) frequencies, with accuracy, stability, and reproducibility as good as the signal sources that we have available. On the way to this long-range goal, however, we will be momentarily content to meet some intermediate goals.

[1] Their papers, which appear elsewhere in these Proceedings of the Seminar on Frequency Standards and Metrology, should be consulted for a more complete picture of progress in infrared frequency synthesis. For a more recent survey of progress in infrared frequency synthesis see Wells et al. 1972.

A. Some Intermediate Goals

The goal of measuring a water vapor laser at 3.8 THz (78 μm), in only one step of multiplication from X-band, has just been achieved a few weeks ago by McDonald et al. 1972. Al Risley has told you about that milestone [or, looking ahead to a unified standard, should we say hertz-stone?]. This step, by a factor of n = 401, is indicated in Figure 1.

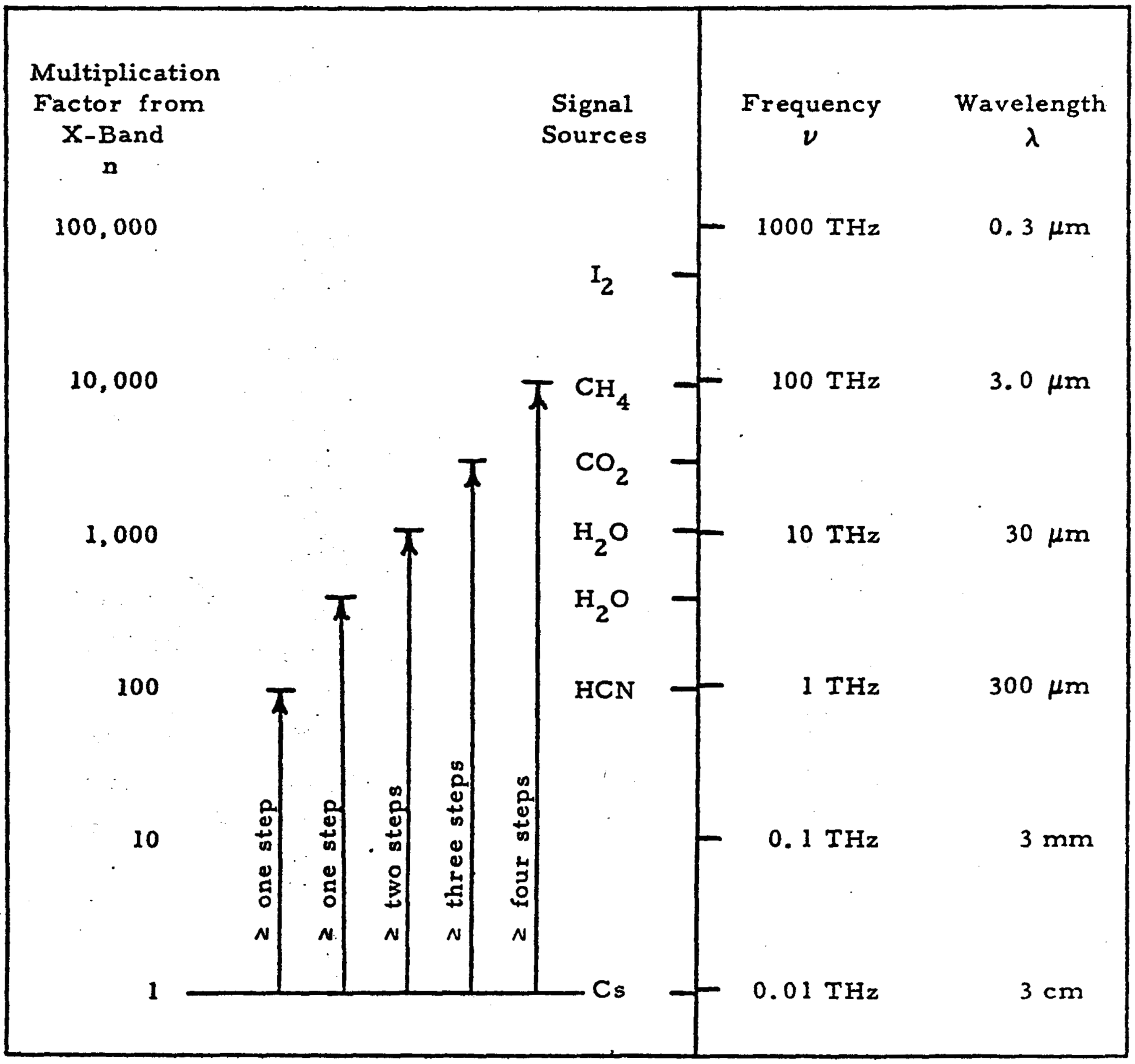

Figure 1. The spectrum from X-band to the visible radiation region. The minimum number of steps which were used in experiments as of end of 1971 are indicated. Multiplication to frequency of I_2 has not been accomplished yet.

The measurement of the HCN laser at 0.89 THz (337 μm) in only one step of multiplication (n = 100) from X-band was achieved a year ago by McDonald et al. 1971. A more difficult, and more significant, goal is to use only one step of multiplication from X-band to measure the frequency of the water vapor laser at 10.7 THz (28 μm). Perhaps it cannot be done, but Don McDonald and Al Risley are sufficiently optimistic about doing it in one step that they are going to spend some of their time trying it. I predict they will succeed.

B. Metrology and Frequency Synthesis

Let us keep in mind that the ability to measure infrared frequencies allows us, in general, through appropriate servos, to control and use these signals for many exacting purposes in metrology. The ability to do the frequency multiplication in the smallest possible number of steps is an exciting goal, for that ability will allow us to reduce cost and inconvenience and to maximize precision, accuracy, and versatility of IR and VR frequency synthesis. With good IR/VR/$\mu\lambda$ frequency synthesis, we will be able to transfer the stability and accuracy of various excellent microwave signal sources [Glaze 1970; Hellwig and Halford 1971] to signals in the IR and VR; conversely, if it develops in the future that the best signal sources are in the IR, then we will be able to transfer their stability and accuracy to the other portions of the frequency spectrum. It is common to use 5-MHz carriers as a "working-frequency" for frequency metrology, to use 1-second ticks for time-scale metrology, and to use visible light for length metrology. I note that these preferences can be maintained, and yet, if desired, stability and accuracy could be improved by frequency synthesis from that region of the spectrum where the best quality signal source exists. There is no compelling need to choose our primary standard so that its frequency is at the preferred working frequency of routine metrology.

C. A Major Goal: Methane

If one-step multiplication from X-band to 10.7 THz proves to be feasible by some means, then we could try one-step multiplication by a factor of about three thousand from X-band to the various frequencies of CO_2 lasers near 29 THz. After that, it would be another factor of about three up to the frequency of the saturated-absorption methane cell frequency standard developed by Barger and Hall 1969a.

At the moment, and for at least the past two years, the methane device has been by far the most stable and accurate signal source in the IR and VR [Barger and Hall 1969a; Barger and Hall 1969b; Hellwig et al. 1972]. Although its present accuracy of about one part in 10^{11} is considerably less than the accuracy for the microwave cesium beam, its stability in the millisecond to one-second range [Hellwig et al. 1972] is unexcelled by any other device. There is considerable promise for further improvements both in accuracy and in stability of the methane device.[2]

Hence, a very significant goal, still unattained by means of frequency metrology as of today (1 September 1971), is the measurement of the frequency of the saturated-absorption methane cell frequency standard at 88 THz (3.39 μm).[3] Full success in the accurate measurement of the frequency of this ultra-stable frequency source will yield[4] an extremely

[2] I note that lasers, per se, generally are no better in stability than are microwave klystrons, per se. In the methane device, as in the cesium beam device, the outstanding stability and accuracy arise not from the slave oscillator but from the passive methane and cesium resonances, respectively. In these devices, the frequency of an oscillator (laser, klystron, quartz crystal, Gunn) is servoed (slaved) to the frequency of a passive resonance by means of frequency metrology. For a discussion of some relevant considerations, see Hellwig 1970.

[3] The first measurement of the 88-THz methane frequency (in terms of the cesium beam frequency standard) by frequency metrology did occur ten weeks later, on 11 November 1971 [Evenson et al. 1972].

[4] Since the methane device is not de jure the standard of length, this determination of the speed of light also requires measurement of the wavelength of the methane device relative to the krypton-86 length standard, as is being done by Barger 1971 and by Giacomo 1971.

accurate determination of the speed of light, or--using a different language of metrology--it will allow accurate length measurements to be referenced to the cesium beam frequency standard as a primary standard for length, with the use of a defined nominal value for the speed of light--this is the fundamental concept of the unified standard.[5] This would reduce the number of independent base units in the International System of Units (SI) from four to only three (Fig. 2).

TIME	"second"	cesium frequency standard
LENGTH	"meter"	krypton wavelength standard
TEMPERATURE	"kelvin"	water triple-point thermodynamic temperature standard
MASS	"kilogram"	prototype hunk of metal

Figure 2. The present four independent base units. The krypton wavelength standard could be dropped (see text), and the meter defined in terms of the second (just as the hertz is already defined in terms of the second).

The main thrust of this paper is an attempt to give a quantitative treatment of some of the frequency stability considerations which are especially relevant to IR frequency synthesis design. The mathematics appear in the following Section. Earlier in this Seminar, Harry Peters and others made the point that good frequency stability will be important for VR/IR/$\mu\lambda$ frequency synthesis and for the unified standard.

[5] Length metrology is not the only field where frequency metrology is having a fundamental impact. At NBS in Boulder, Bob Kamper and associates have done some absolute temperature measurements in terms of frequency standards and involving frequency metrology [Kamper and Zimmerman 1971]. They measured the frequency noise of a Josephson junction oscillator which was coupled to a resistor immersed in a cryogenic bath. The thermodynamic temperature T is related to the frequency noise via fundamental physical relations involving h, e, and k (Planck's constant, charge of an electron, and Boltzmann's constant, respectively). The best [i.e., most reproducible, most stable, most transportable] secondary standard for DC potential difference (electromotive force, EMF) at present is a Josephson junction referenced to a standard frequency [Finnegan et al. 1971]. Barry Taylor and associates in NBS Gaithersburg are using this method as a working secondary standard for maintaining the USA legal volt. For some additional discussion of the progress and feasibility for a unified standard see Halford et al. 1972.

Obviously I agree, and it is a crucial argument in my talk. In addition, I make the point that, for a given frequency stability performance of the signal sources, the optimum utilization of that stability performance will be correspondingly important.

III. THE FAST LINEWIDTH AND RELATED NOISE CONCEPTS

I will discuss a few simple concepts that possibly will allow easy and effective design of low-noise frequency synthesis systems. By keeping down the quantity of perturbing noises, even very weak signals from inefficient non-linear elements (frequency multipliers) can be used for successful frequency synthesis. Efficiency of harmonic generation is desirable, and should be pursued, but low noise levels are also effective.

A. Sources of Noise

In many cases of practical interest, the noise limitations of the VR/IR/$\mu\lambda$ frequency synthesis arise from the frequency instabilities of the two signal sources between which we are trying to do frequency synthesis (see Figure 3). The multiplication process, although sometimes inefficient, often is relatively stable compared to the fluctuations of the oscillators, that is, the multiplicative phase noise is often negligible. It is important then to analyze the noise properties of the oscillators. If the multiplication process is inefficient, other noise sources also will be significant--the additive noises present in the output stages of the frequency multiplier, in the mixer, and in the first stage of the intermediate frequency (IF) amplifier which amplifies the beat frequency.

In the language of electronics, additive noises are those which arise via a linear superposition of the desired signal and a noise signal. Multiplicative noises are those that arise via a modulation of the desired signal by a noise signal. Thermal (kT) noise and shot noise in a mixer are common examples of additive noise signals. A multiplicative phase noise which is common to most electronic phase-processing devices has been characterized by Halford et al. 1968.

Some discussions of the efficiency of infrared harmonic generation, based on recent results with cryogenic Josephson junctions and with room-temperature metal-metal point contacts, are given in McDonald et al. 1972. Their efficiency appears to vary as n^{-2}, and additive noise becomes important for the larger n values.

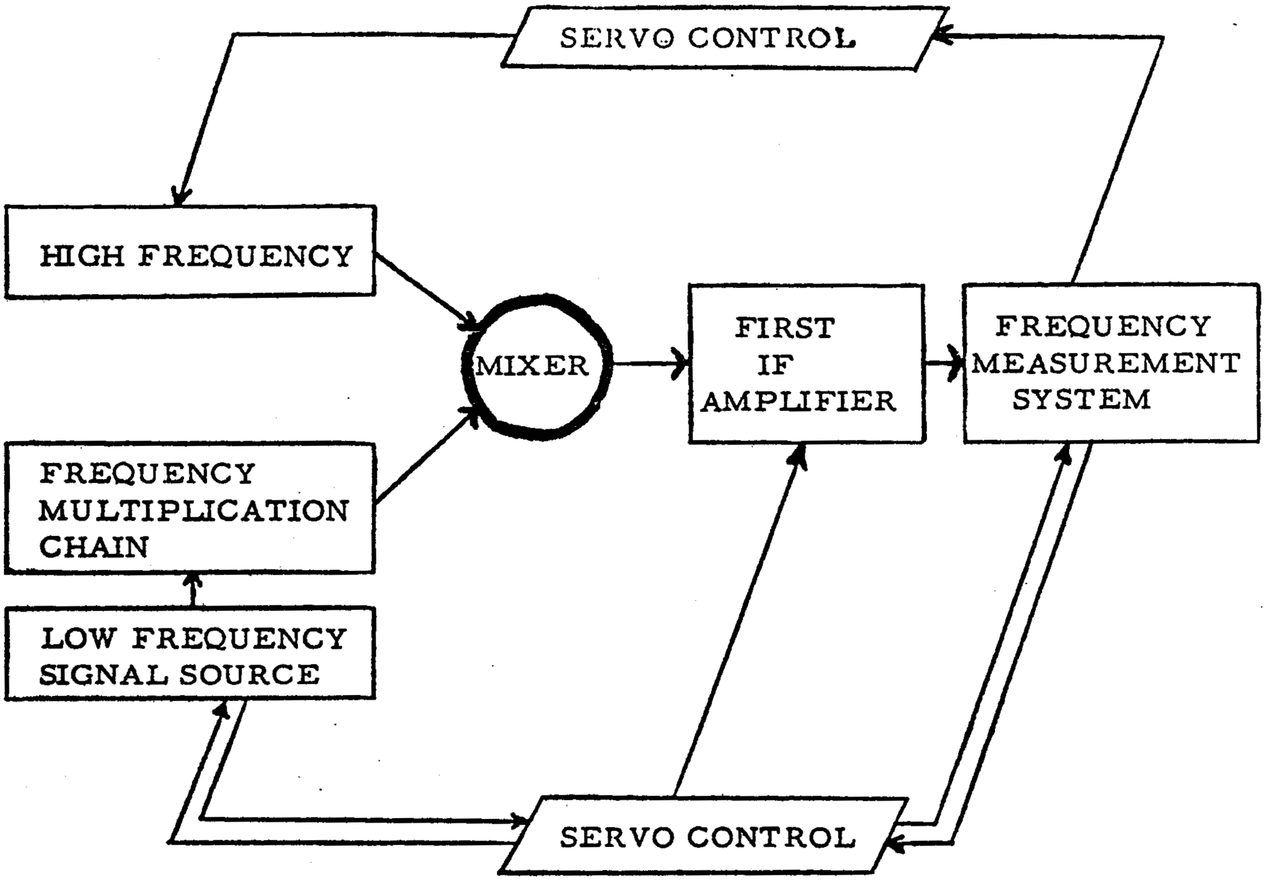

Figure 3. A schematic frequency multiplication/frequency synthesis system. Each of the blocks may be complex, and several servo loops are possible and in general desirable in order to minimize noise problems.

B. Power Law Noise Spectral Densities

Let me take as a constraint that I am always going to consider signal sources (oscillators) that are fairly well behaved. They are the best signal sources that I can buy, borrow, design, or build. I will not be talking about lousy ones, I will be talking about good ones.[6] What we find is that good signal sources will tend to be describable as having some sort of a power law frequency instability over wide ranges of Fourier noise frequency. That means they will have a power law phase instability. And indeed, over the Fourier noise frequency ranges that are going to be important in microwave, infrared, and visible radiation frequency synthesis--they usually do have these pleasing noise behaviors. For some additional discussions of the use of power law spectral densities to characterize high-quality signal sources, see Barnes et al. 1971 and Allan 1966.

The models I am going to discuss will usually assume that linear drift and bright line instabilities either are not present, or else are handled by some additional circuitry or by some additional mathematical treatment not explicitly considered in this paper. These non-random instabilities can be very important and bothersome. They are not emphasized here because their characterization and cure are generally well-understood and are considerably different in nature, as compared to the random noise aspects which are the focus of this paper.

C. The Importance of the Fast Linewidth

When we design and build a frequency multiplication system and expect that the output signal will be very weak, we realize that additive noises will be significant. To minimize the deleterious effects of the additive noises, under the constraint of having a weak signal, we try to reduce the predetection effective noise bandwidth of the measuring system as much as possible. The noise bandwidth is generally set in the IF amplifier.

[6] For some state-of-the-art examples, see Brandenberger et al. 1971 for a quartz crystal oscillator, Glaze 1970 for a multiplier chain with output at X-band, Ashley and Searles 1968 for a two-cavity X-band klystron, and Hellwig et al. 1972 for an 88-THz methane device.

How narrow can we make the noise bandwidth before we begin slicing off significant amounts of the desired signal? If we do not make proper use of various methods of servo control, of various tracking filters, and of programmed filters, then we may have to use a relatively large noise bandwidth to ensure that the desired beat signal is not itself filtered out by the IF amplifier. The large bandwidth will pass a large amount of additive noise, and, if the bandwidth is too large, the desired signal may be swamped by the additive noise.

But suppose we do make appropriate use of servos and dynamic filters. Is there then a limit as to how narrow we can make the noise bandwidth? The answer is yes, there is a limit, and it is greater than zero. The limit is set by what I choose to call the fast linewidth of the beat signal, which is a function of the fast linewidths of the two signal sources which are being compared.

The various properties of the fast linewidth will become apparent later in this paper. Several definitions are possible. The most rigorous treatment which I can offer at the moment, a formal mathematical one, is given in Section III. D. 3.

D. Linewidth Calculations

Some of the noise-linewidth results which I wish to derive and discuss are shown in Fig. 4. A glossary of symbols is given in the Appendix. In general, I have chosen the symbols, noise measures, and language to be similar to the choice used by Barnes et al. 1971.

In this manuscript I do not explicitly treat the cases of power law spectral densities corresponding to white phase noise ($\alpha = +2$) and flicker phase noise ($\alpha = +1$). They have anomalous behavior as compared to the behavior of the cases for $\alpha < +1$. Strictly speaking, their fast linewidths are either zero for zero noise level or are infinite for a non-zero noise level. If a high frequency cut-off is introduced, then a finite fast linewidth can exist. I will treat this important problem elsewhere.

Name of noise	Defining equation	Dependence of fast linewidth upon multiplication factor
White FM	$S_{\delta\nu}(f) \equiv H_0 f^0$	W_f varies as n^2
Flicker FM	$S_{\delta\nu}(f) \equiv H_{-1} f^{-1}$	W_f varies as n^1
Random Walk FM	$S_{\delta\nu}(f) \equiv H_{-2} f^{-2}$	W_f varies as $n^{2/3}$

Figure 4. Frequency modulation (FM) noises of different power law spectral densities, $S_{\delta\nu}$, have different dependences of the fast linewidth W_f upon multiplication factor n. The n-squared dependence for white frequency noise is a classical result. Equations for the fast linewidth are derived in Section III and are listed in Figure 5.

There are several different, but related, ways of reasoning which lead to estimates of the fast linewidth which are sufficiently good for most engineering purposes. I believe it is instructive to illustrate three of these approaches in turn.

1. One radian rms in the time domain. Consider the Allan variance [Allan 1966; Barnes et al. 1971] of the frequency fluctuations of the signal source for

$N = 2,$

$T = \tau,$

τ = variable sample time interval, and

B = bandwidth of the measuring system.

Note that

$$\phi = 2\pi\tau\nu, \tag{1}$$

$$\frac{d\phi}{d\nu} = 2\pi\tau . \tag{2}$$

A measure of mean square phase divergence can be written as

$$(\delta\phi)^2 \quad \left(\frac{d\phi}{d\nu}\right)^2 \sigma^2_{\delta\nu}(N = 2, T = \tau, \tau, B) . \tag{3}$$

If we identify τ^* as that value of τ for which $\delta\phi$ rms is one radian, then

$$(2\pi\tau^*)^2 \ \sigma^2_{\delta\nu}(N = 2, T = \tau^*, \tau^*, B) = 1 \ \mathrm{rad}^2 \ . \tag{4}$$

Define a linewidth W^* such that

$$W^* \equiv \frac{1}{2\pi\tau^*} \ , \tag{5}$$

then

$$W^* = \sigma_{\delta\nu}(N = 2, T = \tau^*, \tau^*, B) \ . \tag{6}$$

I claim (and to me it is intuitively obvious[7]) that W^* is a reasonable approximation to the fast linewidth of a signal at the 3 dB down points, i.e.,

$$W_{f, -3\,\mathrm{dB}} \approx W^* \ . \tag{7}$$

We will see later that indeed this is a good approximation for spectral densities which have the power law form

$$S_{\delta\nu}(f) = H_\alpha f^\alpha \tag{8a}$$

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[7] We may regard τ^* as being a time-domain measure of <u>coherence time</u> of the noisy signal, that is, the time interval required for the signal to become, statistically speaking, about one radian rms out of phase with itself. In length metrology, the corresponding spatial-domain concept is the <u>coherence length</u>, that is, the spatial distance over which the propagated signal becomes about one radian rms out of phase with itself. We may regard $\ell^* \equiv c\tau^*$ as being a measure of coherence length. In a third case--the frequency domain--we can consider the <u>coherence rate</u> of the signal, that is, the rate at which the signal, statistically speaking, becomes about one radian rms out of phase with itself. We may regard $W^* \equiv 1/(2\pi\tau^*)$ as being a measure of coherence rate. (Later I will introduce $W^\dagger$ which may be regarded as being another possible measure of coherence rate.) The fact that the spectral line is 3 dB down at a certain separation from the center of the line (i.e., at $\frac{1}{2} W_{-3\,\mathrm{dB}}$ separation) and is even further down at greater separations, may be regarded as a manifestation of destructive self-interference of those modulation sidebands.

for

$$\alpha \lesssim +0.5 . \tag{8b}$$

From equations (5) - (7), we can say: An approximate value for the fast linewidth $W_{f, -3\,dB}$ of a signal with a power law noise spectral density is

$$W_{f, -3\,dB} \approx \frac{1}{2\pi\tau^*} \tag{9}$$

where τ^* is the solution of the implicit equation

$$\sigma_{\delta\nu}(N = 2, T = \tau^*, \tau^*, B) = \frac{1}{2\pi\tau^*} . \tag{10}$$

Barnes et al. 1971 tabulates[8] some useful relations for converting between frequency domain performance measures and time domain performance measures. For white frequency noise (i.e., $\alpha = 0$)

$$\sigma^2_{\delta\nu}(N = 2, T = \tau^*, \tau^*, B) = \frac{H_0}{2\tau^*} . \tag{11}$$

Combining (11) and (10),

$$\frac{H_0}{2\tau^*} = \left(\frac{1}{2\pi\,\tau^*}\right)^2 = \left(\frac{1}{2\pi\,\tau^*}\right) W^* , \tag{12}$$

$$W^* = \pi H_0 . \tag{13}$$

It is a well-known result for a signal with white frequency noise that the shape of the RF power spectral density curve is a Lorentzian, and it is a well-known classical result that the width at the 3 dB down points is equal to πH_0, i.e.,

[8] The tables in the NBS Technical Note 394 version of Barnes et al. 1971 are more legible than they are in the IEEE Transactions on Instrumentation and Measurement, due to poor typesetting in the latter. Copies of the Technical Note 394 and of a legible conversion table are available upon request.

$$W_{\mathrm{Lorentzian},\,-3\,\mathrm{dB}} = \pi H_0 = \pi S_{\delta\nu} \; . \qquad (14)$$

In the case of white frequency noise, my concept of the fast linewidth and the classical (i.e., traditional) concept of linewidth are identical.

The approximate solution for the fast linewidth given by equations (9) and (10) turns out to be exact for the special case of white frequency noise, but it is not quite exact for other power law spectral densities. Sections III. D. 3 and III. D. 6 give the more carefully derived results and their comparison with these approximations.

2. One radian rms in the frequency domain. Consider the spectral density of the frequency fluctuations of the signal source. The spectral density of the phase fluctuations is related exactly by the factor f^{-2} square radians, i.e.,

$$S_{\delta\phi}(f) = (1\ \mathrm{rad}^2)\,\frac{1}{f^2}\,S_{\delta\nu}(f) \; . \qquad (15)$$

Define a linewidth $W^{\dagger}$ such that the integral of the phase noise spectral density for all Fourier noise frequencies greater than $\frac{1}{2}W^{\dagger}$ amounts to one radian mean square, i.e.,

$$\int_{\frac{1}{2}W^{\dagger}}^{\infty} S_{\delta\phi}(f)\,df = 1\ \mathrm{rad}^2 \; . \qquad (16)$$

I claim (and to me it is intuitively obvious[7]) that $W^{\dagger}$ is a reasonable approximation to the fast linewidth of a signal at the 3 dB down points, i.e.,

$$W_{f,\,-3\,\mathrm{dB}} \approx W^{\dagger} \; . \qquad (17)$$

We will see later that indeed this is a good approximation for power law spectral densities given by equations (8a) and (8b).

The approximate solution given by equations (16) and (17) is not exact for the case of white frequency noise, but it would be exact if in equation (16) we were to require that the integral be set equal to $2/\pi$ square radians instead of being set equal to 1 square radian.

3. Modified Lorentzians of various orders: The width and shape of the complete fast RF power spectral line. In this third, more detailed derivation, I will make a plausible assumption about the shape of the complete fast RF power spectral line. I will provide some justification for the assumption, and then I will make some calculations of fast linewidths that are exact relative to the assumption.

As already mentioned, for the case of a signal having white frequency noise, its RF power spectral line has a Lorentzian shape,[9] i.e.,

$$S_{\sqrt{\text{RF Power}}}(\nu) = a(\alpha = 0)\left\{\frac{1}{1 + \left[\frac{\nu - \nu_0}{\frac{1}{2}W_{-3\,\text{dB}}}\right]^2}\right\}, \tag{18}$$

where, for the narrowband approximation (i.e., that ν_0 is much greater than $W_{-3\,\text{dB}}$),

$$a(\alpha = 0) = \frac{1}{\pi}\left(\frac{2}{W_{-3\,\text{dB}}}\right)P, \tag{19}$$

$$P \equiv \int_0^\infty \left\{S_{\sqrt{\text{RF Power}}}(\nu)\right\} d\nu \,. \tag{20}$$

By studying the behavior of Bessel functions, and applying that knowledge to the problem of relating phase modulation to sideband power, we can easily decide some simple relations. We find for those sideband components

[9] In keeping with the constraint that I am considering only good quality signal sources, I explicitly am assuming that there are no amplitude modulation (AM) sidebands.

for which $|\nu - \nu_0|$ is much greater than the fast linewidth,[10] i.e.,

$$|\nu - \nu_0| \gg W_{f,-3\,dB}, \tag{21}$$

that the sideband power is proportional to the square of the modulation index, that is, it is proportional to the square of the phase modulation intensity. In a language which is more suitable for discussing random noise, we find under the condition expressed by equation (21) that the fast RF power spectral density[11] at $|\nu - \nu_0|$ from the center of the fast line is proportional to the spectral density[11] of the phase noise at the Fourier noise frequency f, with f equal to $|\nu - \nu_0|$.

$$S_{\sqrt{\text{RF Power}}}(\nu_0 \pm f) = b\,P\,S_{\delta\phi}(f), \tag{22}$$

$$f = |\nu - \nu_0|, \tag{23}$$

$$f \gg W_{f,-3\,dB}, \tag{24}$$

and

$$b = \text{constant} = \frac{1}{2}\,\text{rad}^{-2}. \tag{25}$$

That the coefficient b is equal to $\frac{1}{2}\,\text{rad}^{-2}$ can be derived using Bessel functions, but it can also be derived from even simpler trigonometric identities.

Combining equations (22) and (15) we see for

$$S_{\delta\nu}(f) = H_\alpha f^\alpha \tag{26}$$

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[10] This is equivalent to the condition that the modulation index be much smaller than unity, i.e., that the phase modulation is much less than one radian--for the components under consideration.

[11] Please note that as in Barnes et al. 1971, I use one-sided spectral densities. In equation (22) I write the argument of the fast RF power spectral density as $(\nu_0 \pm f)$ to indicate and to remind us that, in the concept of the fast RF spectral line, the center of the fast line, ν_0, is changing with time in the general case.

that

$$S_{\sqrt{\text{RF Power}}}(\nu_o \pm f) = \frac{1}{2} P H_\alpha f^{\alpha-2} \quad , \tag{27}$$

for

$$f >> W_{f,-3\,dB} \; . \tag{28}$$

We note that this is fully in agreement with equations (18) and (19).

I now make an assumption about the shape of the fast RF power spectral lines for power law spectral densities of frequency noise for cases other than white frequency noise. Based on the insight gained in the previous discussion, I assume that for all

$$\alpha < +1. \tag{29}$$

that

$$S_{\sqrt{\text{RF Power}}}(\nu_0 \pm f) = a(\alpha) \left\{ \frac{1}{1 + \left[\dfrac{\nu - \nu_0}{\frac{1}{2} W_{f,-3\,dB}} \right]^{2-\alpha}} \right\} . \tag{30}$$

The shape function in equation (30) is a Lorentzian for α equal to zero. We may call it a "modified Lorentzian of order $(1 - \frac{1}{2}\alpha)$" for the general case.

I do not know whether or not equation (30) is in detail the correct expression for the fast RF power spectral line. From the insight given by equations (18) through (27), equation (30) is a plausible guess, and, even if incorrect, surely it cannot be very far from being correct. From equations (27) and (25) we see that equation (30) is correct for the wings of the fast line, and the only possible discrepancy would be in the shape at and near the top of the fast line. For the power law spectral density of frequency noise cases which are under consideration, I do not expect any "structure" [other than the smooth form of equation (30)] would exist in the shape at and near the top of the fast line.

Hopefully, someone soon will devise a proper derivation of equation (30). In the meantime, I assume it is correct, and I use it to calculate the fast linewidth for general $\alpha < +1$.

It is convenient to consider the normalized fast RF power spectral density associated with phase fluctuations, and I use the symbol script $\mathscr{L}(f)$ for that quantity [Glaze 1970; Meyer 1970a; Meyer 1970b].

$$\mathscr{L}(f) \equiv \frac{S_{\sqrt{\text{RF Power}}}(\nu_0 \pm f)}{P} . \tag{31}$$

From equations (22)-(26), (15), and (31) we see that we may write

$$\mathscr{L}(f) = \left(\frac{1}{2}\,\text{rad}^{-2}\right) S_{\delta\phi}(f), \tag{32}$$

$$\mathscr{L}(f) = \frac{1}{2}\left(\frac{1}{f^2}\right) S_{\delta\nu}(f) , \tag{33}$$

$$\mathscr{L}(f) = \frac{1}{2}\, H_\alpha\, f^{\alpha - 2} , \tag{34}$$

for

$$f \gg W_{f,\,-3\,\text{dB}} . \tag{24}$$

Combining equations (30) and (20),

$$P = a(\alpha) \int_0^\infty \left\{ 1 + \left[\frac{\nu - \nu_0}{\frac{1}{2} W_{f,\,-3\,\text{dB}}}\right]^{2-\alpha} \right\}^{-1} d\nu . \tag{35}$$

With use of the narrowband approximation, we find that the integral in equation (31) may be evaluated, for all $\alpha < +1$, to give

$$P = \left[a(\alpha)\right] \left(\frac{\pi}{2 - \alpha}\right) \left[\sin \frac{\pi}{2 - \alpha}\right]^{-1} W_{f,\,-3\,\text{dB}} . \tag{36}$$

Combining equations (36), (31), and (30), and under the constraint of equation (24)

$$f \gg W_{f,\,-3\,\mathrm{dB}}\;, \tag{24}$$

we may write

$$\mathcal{L}(f) = \frac{1}{2}\left(\frac{\pi}{2-\alpha}\right)^{-1}\left(\sin\frac{\pi}{2-\alpha}\right)\left(\tfrac{1}{2}W_{f,\,-3\,\mathrm{dB}}\right)^{1-\alpha} f^{\alpha-2}. \tag{37}$$

We now combine equations (37) and (34) and solve for the fast linewidth,

$$W_{f,\,-3\,\mathrm{dB}} = 2\left\{\frac{\left(\frac{\pi}{2-\alpha}\right)H_\alpha}{\sin\frac{\pi}{2-\alpha}}\right\}^{\frac{1}{1-\alpha}}. \tag{38}$$

See Figure 5 for some solutions of equation (38) for some special cases.

4. Some properties of the fast linewidth. Frequency multiplication can also be described as being phase angle multiplication or phase angle amplification.[12] We may write

$$\delta\hat{\phi} = n\,\delta\phi\;, \tag{39}$$

$$\delta\hat{\nu} = n\,\delta\nu\;, \tag{40}$$

$$S_{\delta\hat{\phi}}(f) = n^2 S_{\delta\phi}(f)\;, \tag{41}$$

$$S_{\delta\hat{\nu}}(f) = n^2 S_{\delta\nu}(f)\;, \tag{42}$$

and

$$\sigma_{\hat{y}}^2(\tau) = \sigma_y^2(\tau)\;, \tag{43}$$

[12] Indeed, the engineering design of a multiplier chain for good noise performance is in many ways analogous to the design of a low-noise amplifier. In an optimum design, the phase noise of a phase amplifier (frequency multiplier) is set by the input stage and the input stage is carefully designed to maximize its performance. Later stages tend to not contribute significant quantities of phase noise relatively speaking, because their noise tends to be swamped by the amplified phase noise of the preceding stages. This design aspect is fully valid at the lower frequencies [e.g., see Glaze 1970 and Meyer 1970a], and it may be expected to become more and more valid at the higher (IR/VR) frequencies.

α	$S_{\delta\nu}(f) = H_\alpha f^\alpha$	Fast Linewidth, $W_{f,-3\,dB}$
+0.9	$H_{0.9} f^{0.9}$	$2\left(\frac{\pi}{1.1 \sin \frac{\pi}{1.1}}\right)^{10} \left(H_{0.9}\right)^{10} = 2.2 \times 10^{10} \left[S_{\delta\nu}(1)\right]^{10}$
+0.5	$H_{0.5} f^{0.5}$	$\frac{32}{27} \pi^2 \left(H_{0.5}\right)^2 = 1.2 \times 10^1 \left[S_{\delta\nu}(1)\right]^2$
0	$H_0 f^0$	$\pi H_0 = 3.14 \left[S_{\delta\nu}(1)\right]$
-1	$H_{-1} f^{-1}$	$\left(\frac{8\pi}{3\sqrt{3}}\right)^{\frac{1}{2}} \left(H_{-1}\right)^{\frac{1}{2}} = 2.20 \left[S_{\delta\nu}(1)\right]^{\frac{1}{2}}$
-2	$H_{-2} f^{-2}$	$\left(2\sqrt{2}\,\pi\right)^{\frac{1}{3}} \left(H_{-2}\right)^{\frac{1}{3}} = 2.08 \left[S_{\delta\nu}(1)\right]^{\frac{1}{3}}$
-3	$H_{-3} f^{-3}$	$\left(\frac{16\pi}{5 \sin \frac{\pi}{5}}\right)^{\frac{1}{4}} \left(H_{-3}\right)^{\frac{1}{4}} = 2.04 \left[S_{\delta\nu}(1)\right]^{\frac{1}{4}}$

Figure 5. A tabulation of fast linewidths given by equation (38) for some power law spectral densities of common interest. Note for $\alpha \ll 0$ that the fast half-linewidth is approaching the $(1 - \alpha)$th root of the spectral density of the frequency fluctuations evaluated at unity frequency. Equation (38) is not valid for $\alpha \geq 1$. Equation (38) and Figure 5 are valid in any self-consistent set of units. Note that $H_\alpha = S_{\delta\nu}(1)$.

where $\hat{\phi}$, $\hat{\nu}$, $\hat{y}$ are the output phase, output frequency, and output fractional frequency fluctuations, respectively, of a multiplier chain characterized by a multiplication factor (phase amplification factor) of n. The same symbols without carets (^) are the respective input quantities.

From equations (42) and (8a), we may write

$$\hat{H}_\alpha = n^2 H_\alpha . \tag{44}$$

Combining equations (44) and (38) we obtain the dependence of the fast linewidth of a signal as the signal is frequency-multiplied in a frequency multiplier. The frequency multiplier is assumed to be noiseless in the present discussion.

$$\hat{W}_{f,-3\,dB} \approx \left[\hat{H}_\alpha\right]^{\frac{1}{1-\alpha}} , \tag{45}$$

$$\hat{W}_{f,-3\,dB} \approx n^{\frac{2}{1-\alpha}} \left[H_\alpha\right]^{\frac{1}{1-\alpha}} . \tag{46}$$

This dependence of the fast linewidth upon n is shown in Figure 4 for three common values of α.

It is amusing to note that the fast linewidth of a flicker frequency noise signal does not blow up with multiplication factor as strongly as does the linewidth of a white frequency noise signal. The dependences are as n and n^2, respectively. In multiplying from the frequency of a cesium beam (10^{10} Hz) to the frequency of a carbon dioxide laser (3×10^{13} Hz), the factor by which the white noise signal would broaden would be three thousand times greater than the factor for the flicker noise signal.[13]

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[13] This does not mean we should hope for flicker noise rather than for white noise--there are other considerations!

Another point of interest is that the fast linewidth of a flicker frequency noise signal is proportional to the multiplication factor. This behavior is unique to the flicker noise case.[14] Of the various power law noises, only for flicker of frequency noise is the fractional linewidth of the frequency-multiplied signal independent of the multiplication factor.

5. A graphical fast linewidth construction. It is possible to make a graphical construction which gives a solution for the fast linewidth. See Figure 6. Although only approximate, this graphical solution shows in a revealing fashion the dependence of W_f upon the random noise level of the signal source (e.g., a microwave oscillator), the dependence of W_f upon the multiplication factor n, and the general way in which different power laws for the phase noise spectral density affect W_f. In Figure 6 the "W locus" is drawn so that it passes through the point

$$f = 1 \text{ Hz}; \quad S_{\delta\phi}(f) = -7 \text{ dB relative to } 1 \text{ rad}^2 \text{ Hz}^{-1} \quad ,$$

and so that it has a slope corresponding to f^{-1}. The phase noise spectral density, $S_{\delta\phi}(f)$, of the output of the oscillator is multiplied by n^2 (to give the random phase noise at the n-th harmonic) and plotted. At some highest frequency, f_x, the random noise plot will cross the W locus and be below it for all higher f. Simply identify f_x as W_f and for typical random noises encountered in good oscillators, this approximate value for W_f will be correct within a factor of two or thereabouts. This is probably adequate for preliminary engineering design of the frequency synthesis systems of present interest.

The intercept of -7 dB is chosen to be correct for flicker of frequency noise ($\alpha = -1$), and it is a good approximation for the other commonly-encountered noise laws. For example, to be exact the W locus intercept would be only 2 dB higher for white FM ($\alpha = 0$) and would be only 2.5 dB lower for random walk FM noise.

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[14] The case of flicker frequency noise ($\alpha = -1$) is common, and its simple behavior under frequency multiplication allows some of the usual calculations concerning noise to be done easily in one's head.

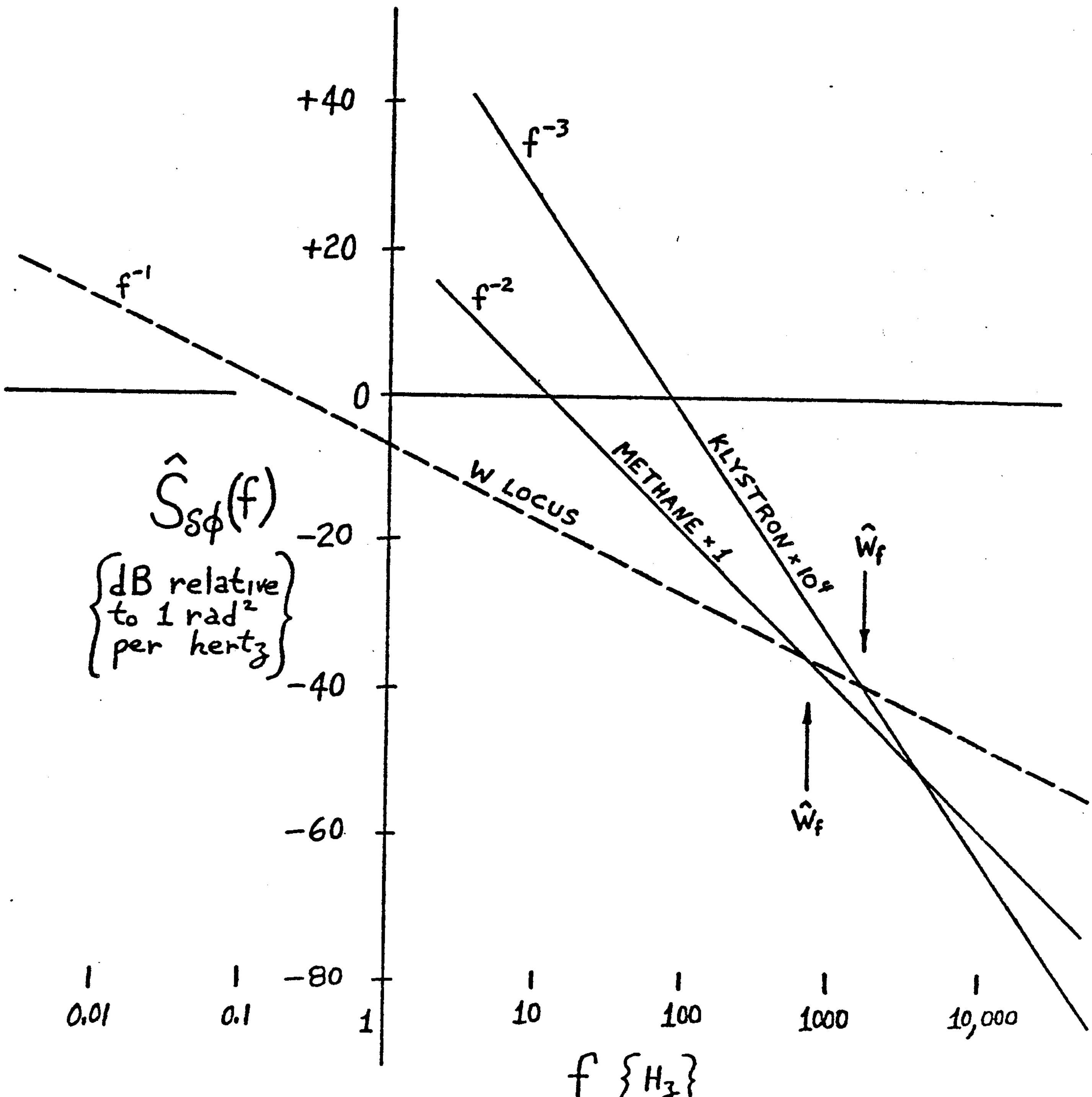

Figure 6. Graphical construction for solving and understanding fast linewidth problems. Examples given are hypothetical (see text).

In Figure 6 I consider the hypothetical case of an X-band klystron signal multiplied by a factor n of 10^4 up to the frequency of the methane device. I assume that the klystron signal has a flicker of frequency noise of one part in 10^{11}. That is, the Allan variance for N = 2, T = τ, τ, and B is

$$\sigma_y^2(\tau) = \left(1 \times 10^{-11}\right)^2 \tau^0 . \tag{47}$$

If multiplied to 88 THz, the phase noise would be

$$\hat{S}_{\delta\phi}(f) = (1 \text{ rad}^2)(\hat{\nu}_0)^2 \left[\frac{\sigma_y^2(\tau)}{2 \ln 2} \right] f^{-3} , \tag{48}$$

$$\hat{S}_{\delta\phi}(f) = (5.7 \times 10^5 \text{ rad}^2 \text{ Hz}^2)\, f^{-3} , \tag{49}$$

and

$$n \approx 10^4 . \tag{50}$$

I also consider the hypothetical case of a methane device with direct output at 88 THz. I assume the methane device has a white frequency noise with a one-second stability of one part in 10^{13}. That is, the Allan variance for N = 2, T = τ, τ, and B is

$$\sigma_y^2(\tau) = \left(1 \times 10^{-13}\, s^{\frac{1}{2}}\right)^2 \tau^{-1} . \tag{51}$$

The phase noise of this methane signal is

$$\hat{S}_{\delta\phi}(f) = (1 \text{ rad}^2)(\hat{\nu}_0)^2 \left[2\tau\, \sigma_y^2(\tau) \right] f^{-2} , \tag{52}$$

$$\hat{S}_{\delta\phi}(f) = (1.6 \times 10^2 \text{ rad}^2 \text{ Hz})\, f^{-2} , \tag{53}$$

and

$$n = 1 . \tag{54}$$

Equations (49) and (52) are plotted in Figure 6. From their intersection with the W locus we obtain the fast linewidths for the methane signal and the multiplied klystron signal to be 680 Hz and 1700 Hz, respectively. Using equation (38) [or Figure 5] the fast linewidths are calculated to be 500 Hz and 1700 Hz, respectively.

Since multiplication by a factor of ten, for example, would correspond to a shift of 20 dB on Figure 6, one can look at Figure 6 and quickly comprehend how the various fast linewidths would change if the multiplication factor were changed.

Although the discussion so far has been for pure power law spectral densities, the graphical construction of Figure 6 can be used to estimate the fast linewidth of signals whose spectral behavior is not as simple as a pure power law.

6. Utility of W^* and of $W^\dagger$ as approximations to the fast linewidth W_f. The methods of calculation of W^* and $W^\dagger$ involve the usual frequency/time stability measures, and do not involve any measures of radio frequency power. This simplification makes them easy to use in noise calculations. If W^* and $W^\dagger$ are adequate approximations to the fast linewidth W_f, then we have an easy way to introduce the (approximate) fast linewidth in noise calculations without being required to use explicitly RF power spectral density mathematics. In Figure 7, I show a comparison of W^*, $W^\dagger$, and W_f for three common random frequency noise power law spectral densities. We can see that the approximation is sufficiently good for many engineering purposes.

Frequency Noise Spectral Density $S_{\delta\nu}(f) = H_\alpha f^\alpha$	Phase Noise Spectral Density $S_{\delta\phi}(f) = (1\ \text{rad}^2) H_\alpha f^{\alpha-2}$	Allan Variance of Frequency Fluctuations for $N = 2$, $T = \tau$, τ, B $\sigma^2_{\delta\nu}(\tau)$	$W*$	$W^\dagger$	W_f
"White FM" $\alpha = 0$ $H_0 f^0$	$(1\ \text{rad}^2) H_0 f^{-2}$	$(2\tau)^{-1} H_0$	πH_0	$2 H_0$	πH_0
"Flicker FM" $\alpha = -1$ $H_{-1} f^{-1}$	$(1\ \text{rad}^2) H_{-1} f^{-3}$	$(2 \ln 2) H_{-1}$	$\left[(2 \ln 2) H_{-1}\right]^{\frac{1}{2}}$	$\left[2 H_{-1}\right]^{\frac{1}{2}}$	$\left[\left(\frac{8\pi}{3\sqrt{3}}\right) H_{-1}\right]^{\frac{1}{2}}$
"Random Walk FM" $\alpha = -2$ $H_{-2} f^{-2}$	$(1\ \text{rad}^2) H_{-2} f^{-4}$	$\left(\frac{2}{3}\pi^2\tau\right) H_{-2}$	$\left[\left(\frac{\pi}{3}\right) H_{-2}\right]^{\frac{1}{3}}$	$\left[\frac{8}{3} H_{-2}\right]^{\frac{1}{3}}$	$\left[(2\sqrt{2}\pi) H_{-2}\right]^{\frac{1}{3}}$

Figure 7. A comparison of $W*$, $W^\dagger$, and W_f for three common random noise power law spectral densities. The entries for $W*$, $W^\dagger$, and W_f are calculated from equations (6), (16), and (38) [or see Figure 5], respectively. The values for the Allan variance are from the tabulation of Barnes et al. 1971.

IV. DISCUSSION

If the instabilities of all signal sources were merely white frequency noise, then the mathematics of Section III would not be very relevant to infrared-microwave frequency synthesis. There would be no need for the concept of the fast linewidth, for the linewidth of the signal would be independent of the time interval we use to look at the signal.

Conversely, if the spectral density of the frequency fluctuations is low-frequency divergent, then the longer the time interval τ we use to observe the radio-frequency power spectral linewidths $W_{-3\,dB}(\tau)$, the broader that linewidth will be. This is the result of the moving around in frequency of the fast line. In principle, and in practice, it is possible to build a servo to tune a superheterodyne receiver having a narrowband IF amplifier to track such a moving signal. The bandwidth of the IF amplifier must be larger than the fast linewidth of the signal, but it can be narrowed down toward that bandwidth as a limit.

The stabilities of available X-band signal sources and 88-THz methane devices are sufficiently good that we can conceive of frequency synthesis designs to connect X-band with 88 THz which could have IF amplifier noise bandwidths in the 10^3 Hz range. See Figure 3. With the improvements which are occurring in the stability of these signal sources, we can conceive of optimum designs for the future in which the required noise bandwidth might get down to the 10^2 Hz range.

In order to successfully utilize such a narrowband system, several control systems will generally be needed, and special auxiliary filters will be needed. For example, it usually will be necessary to use passive filters on the output of an X-band oscillator to filter off noise sidebands and spurious signals which are hundreds of kilohertz from the center frequency ("carrier"). The second-to-second instability of the X-band oscillator probably must be reduced by servoing it to the output of a multiplier chain driven by a state-of-the-art quartz crystal oscillator. The hour-to-hour (and longer) instability probably must be reduced by servoing this X-band system to a microwave atomic frequency standard, e. g., a commercial rubidium gas cell device.

Instead of actually servoing the frequency of the X-band oscillator, it might be easier and more effective to servo the center frequency of one of the IF amplifiers, as indicated in Figure 3.

If an idler oscillator is used in the frequency multiplication chain, e. g., at 10 THz, its relatively large instabilities will require careful application of either or both of the techniques just mentioned. If it is difficult to servo the frequency of the idler oscillator due to poor tuneability, the alternate procedure of controlling the center frequency of an IF amplifier becomes very attractive. In principle, this would allow the idler oscillator to be free-running and yet not contribute to the noise of the measurement. In such a scheme the role of the idler oscillator would be purely to supply the high power needed for good efficiency at an intermediate stage of the frequency multiplication chain.

The point of this filtering and servoing is to reduce the total quantity of additive noise by minimizing the noise bandwidth of the IF amplifier. If the beat signal power is sufficiently greater than the noise power in the narrowband IF bandwidth, it will be possible to do cycle counting by using the zero-crossings of the beat signal. If this can be done, we will be able to achieve very high accuracy in the frequency synthesis, for the uncertainty will tend to be of the ±1-count type.

If cycle counting cannot be done, then we will have to acquire a certain amount of patience and use power spectral density detection techniques, as opposed to zero-crossing techniques, and we will have to average for times long compared to the $(\text{linewidth})^{-1}$ in order to make useful measurements of the beat signal. The accuracy as well as the precision of the frequency comparison will not be as good as for cycle counting, but it can still be very good.

There are many other signal processing tricks which can be employed in the "frequency measurement system" block and in the various servo control systems. If we can achieve greater efficiency of frequency multiplication, then we can reduce the complexity of the apparatus. Conversely, if we are willing to increase the complexity of the apparatus, then we will be able to successfully use non-linear elements of relatively low efficiency. An understanding of the fast linewidth allows us to see more clearly how the trade-off should go.

The behavior of the fast linewidth under multiplication suggests that we should be seriously searching for methods of IR/VR frequency division as a superior alternative to frequency multiplication. If divide-by-two, or divide-by-n, flip-flop circuits (or any other type of frequency division elements) could be achieved at the 88 THz frequency, the beat signal used in Figure 3 could be taken at a lower frequency. The fast linewidth of the beat signal would be narrower than if the beat signal were taken at 88 THz, and the IF bandwidth could be narrower.

The ultimate extension of course would be, if possible, to repeatedly divide all the way down to the lower frequency. I strongly suggest that possibilities for partial or complete IR/VR frequency division be sought, studied, and exploited.

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