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Efficient Numerical and Analog Modeling of Flicker Noise Processes

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Efficient Numerical and Analog Modeling of Flicker Noise Processes

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Efficient Numerical and Analog Modeling of Flicker Noise Processes

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It is shown that by cascading a few simple resistor-capacitor filters, a filter can be constructed which generates from a white noise source a noise signal whose spectral density is very nearly flicker, $|f|^{-1}$, over several decades of frequency f. Using difference equations modeling this filter, recursion relations are obtained which permit very efficient digital computer generation of flicker noise time-series over a similar spectral range. These analog and digital filters may also be viewed as efficient approximations to integrators of order one-half.

Key words: Analog noise simulation; computer noise simulation; digital filters; flicker noise; fractional integration; recursive digital filters.

1. INTRODUCTION

One of the most pervasive noise processes observed in electronic systems is flicker noise, whose spectral density varies as $|f|^{-1}$ over many decades of frequency, often beyond the low frequency limit to which the spectral density can reasonably be measured. (See, for example [1] - [6] and their bibliographies.) While its occurrence is very widespread, its cause, or causes, is as yet uncertain, despite the attention of many investigators.

In order to study flicker noise and to simulate the behavior of systems with flicker noise, several authors have presented mathematical models which generate a discrete [7], [8] or continuous [9] - [14] flicker noise from a white noise process. Because of the intrinsically long correlation time associated with the flicker noise process, the models have required large computer memories, or extensive computation, or both. The purpose of this paper is to present an algorithm which permits one to generate, in a very efficient manner with negligible computer memory requirements, a sequence of numbers with a very nearly flicker noise spectral density over several decades of frequency. It is derived from a cascade of simple resistor-capacitor filters which, when realized, also permits one to generate from a white noise source an analog noise signal which has a spectral density which is very nearly flicker over several decades of frequency.

2. THE FILTER CASCADE

Consider the filter shown in figure 1. Its voltage transfer function is:

$$g(\omega) = \frac{y(\omega)}{x(\omega)} = \frac{1 + j\omega\tau_2}{1 + j\omega(\tau_1 + \tau_2)} \tag{1}$$

where

$$\tau_1 = R_1 C \tag{2}$$

$$\tau_2 = R_2 C \tag{3}$$

and

$$\omega = 2\pi f. \tag{4}$$

Equation (1) is valid provided the loading on the output, y, of the filter is negligible. This assumption, thus, might require the filter to be followed by an isolation amplifier in practice. As it will be seen, however, it is possible to construct a complete flicker filter with only one amplifier and still not compromise the validity of eq (1).

For low frequencies ($\omega(\tau_1 + \tau_2) << 1$), one finds that $g(\omega) \approx 1$; and for high frequencies ($\omega\tau_2 >> 1$), one finds $g(\omega) \approx \frac{\tau_2}{\tau_1 + \tau_2}$. If one were to have a sequence of such filters such that for the i-th filter

$$\tau_1^{(i)} = (\beta)^i \tau_1^{(0)}, \quad i = 1, 2, \ldots N, \tag{5}$$

where β is some constant such that $\beta < 1$,

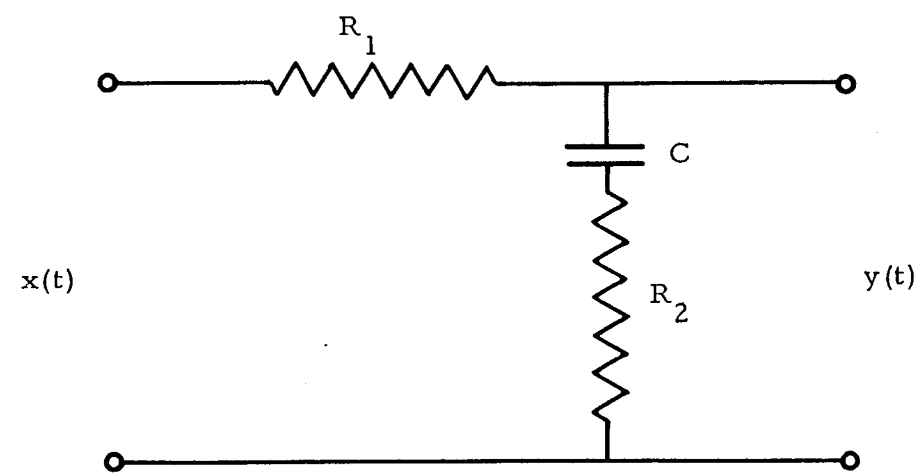

Figure 1. Basic R-C filter with voltage transfer function given by eq (1).

and

$$\tau_2^{(i)} = (\beta)^i \, \tau_2^{(o)}, \tag{6}$$

and if these filters were cascaded with appropriate isolation amplifiers between successive stages (to eliminate loading effects), the overall voltage transfer function would become:

$$G(\omega) = \prod_{i=1}^{N} \left[\frac{1 + j\omega\tau_2^{(o)}\beta^i}{1 + j\omega(\tau_1^{(o)} + \tau_2^{(o)})\beta^i} \right] . \tag{7}$$

One may consider a set of n sections of this filter cascade corresponding to $i = j + 1, j + 2, \ldots j + n$, where j is arbitrary. This subset of filters would change the (magnitude) transfer function by the factor

$$\left(\frac{\tau_2^{(o)}}{\tau_1^{(o)} + \tau_2^{(o)}} \right)^n$$

in the frequency interval

$$\omega_j < \omega < \omega_{j+n}, \tag{8}$$

where

$$\omega_j = \frac{1}{\tau_2^{(i)}} = \frac{1}{\beta^j \, \tau_2^{(o)}} .$$

If this filter is to approximate a filter whose transfer function varies as

$$|G^*(\omega)|^2 = A^2 |\omega|^\alpha, \tag{9}$$

where A is a constant, then

$$\left| \frac{G^*(\omega_{j+n})}{G^*(\omega_j)} \right|^2 = \left| \frac{\omega_{j+n}}{\omega_j} \right|^\alpha = \beta^{-n\alpha} \approx \left(\frac{\tau_2^{(o)}}{\tau_1^{(o)} + \tau_2^{(2)}} \right)^{2n} . \tag{10}$$

Thus,

$$\beta = \left(\frac{\tau_2^{(o)}}{\tau_1^{(o)} + \tau_2^{(o)}} \right)^{-\frac{2}{\alpha}} . \tag{11}$$

For a flicker filter $\alpha = -1$ in eq (8) and, thus, eq (11) becomes

$$\beta = \left(\frac{\tau_2^{(o)}}{\tau_1^{(o)} + \tau_2^{(o)}}\right)^2 . \tag{12}$$

A more precise approximation to a flicker filter is obtained by increasing β (i.e., the ratio of the τ's given in eq (12)). However, the frequency range of the complete filter is proportional to β^{-N}. Thus, to improve the precision without reducing the frequency range both β and the number of sections, N, must be increased.

It is of value to take as a practical example the particular values:

$$N = 4 \tag{13}$$

$$\tau_1^{(o)} + \tau_2^{(o)} = 3\tau_2^{(o)} \tag{14}$$

and use the variable

$$s \equiv j\omega\, \tau_2^{(1)}. \tag{15}$$

The filter's transfer function then becomes (from eq (7))

$$G(\omega) = \left(\frac{s+1}{3s+1}\right)\left(\frac{s+9}{3s+9}\right)\left(\frac{s+81}{3s+81}\right)\left(\frac{s+729}{3s+729}\right). \tag{16}$$

This transfer function should be expected to have approximately an $s^{-\frac{1}{2}}$ behavior over a relative frequency range of $\beta^{-4} = 9^4$, nearly four decades of frequency. In fact, the magnitude error

$$\mathcal{E} \equiv |G(\omega)| - A|s^{-\frac{1}{2}}|, \tag{17}$$

is within $\pm \frac{1}{2}$ dB relative to $A|s^{-\frac{1}{2}}|$ over the range $0.25 \le |s| \le 1000$, a relative frequency range of 4000 where $A = 0.43$. The location of the approximated frequency band depends on $\tau_2^{(1)}$. For the value $\tau_2^{(1)} = 0.04$ seconds, the approximating band is (1 Hz, 4 kHz); the error is shown in figure 2.

One may forget, for the time being, the heuristic arguments which led to generating eq (16) as an approximation to the function

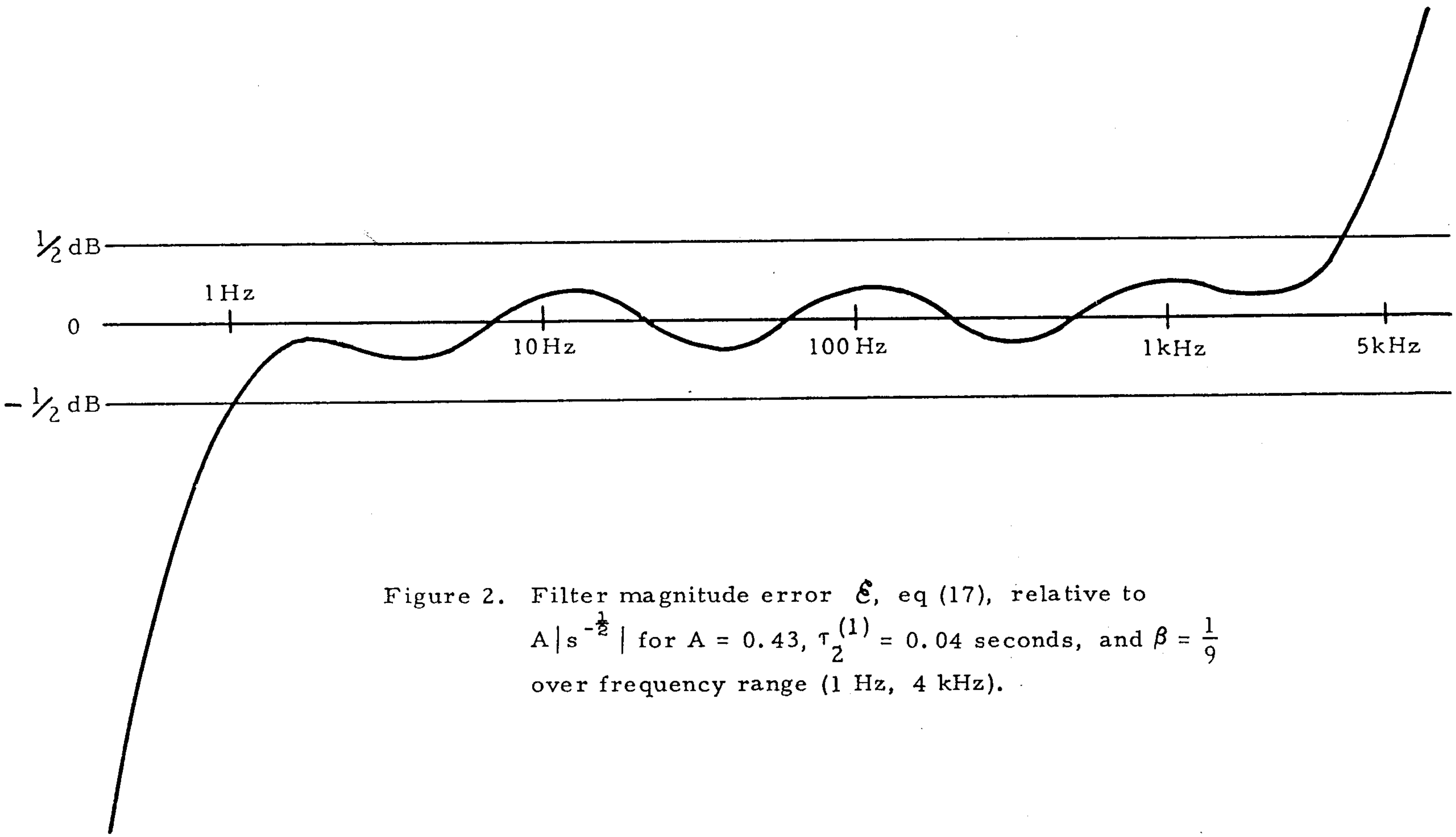

Figure 2. Filter magnitude error $\mathcal{E}$, eq (17), relative to $A|s^{-\frac{1}{2}}|$ for $A = 0.43$, $\tau_2^{(1)} = 0.04$ seconds, and $\beta = \frac{1}{9}$ over frequency range (1 Hz, 4 kHz).

$$G^*(\omega) = A\, s^{-\frac{1}{2}}. \tag{18}$$

Equation (16) simply gives a mathematical function which approximates eq (18) rather well over a substantial frequency range. Thus, one could consider an impedance, $z_1(\omega)$, defined by

$$z_1(\omega) \equiv R_0 G(\omega), \tag{19}$$

where the constant R_0 has the dimensions of ohms. The exact realization of $z_1(\omega)$ may be obtained by a partial fractions expansion of eq (16) and one obtains the impedance shown in figure 3 for $R_0 = 20\text{k}\Omega$ and $\tau_2^{(1)} = 0.04$ seconds. (The exact realization of an impedance given by eq (18) would be much more difficult.) Thus, figure 3 is an approximation to a "fractional capacitor" [15] - [17] of order one-half.

This impedance can be used in a straightforward fashion in connection with a single operational amplifier to realize a transfer function given by eq (16). It is no longer necessary to consider the filter cascade with its isolation amplifiers mentioned above. This new, active filter is realized as follows:

An operational amplifier is a device whose transfer function is approximated by a large, real, negative number, -k, over a large frequency range which is assumed to include the frequency domain of interest. If one connects an operational amplifier with two impedances $z_1(\omega)$ and $z_2(\omega)$ as shown in figure 4, the overall transfer function becomes

$$\frac{e_0(\omega)}{e_i(\omega)} = \frac{-k}{1 + (1+k)\, z_2/z_1} . \tag{20}$$

When

$$k >> \left|\frac{z_1}{z_2}\right|,$$

then

$$\frac{e_0(\omega)}{e_i(\omega)} \approx \frac{z_1}{z_2} . \tag{21}$$

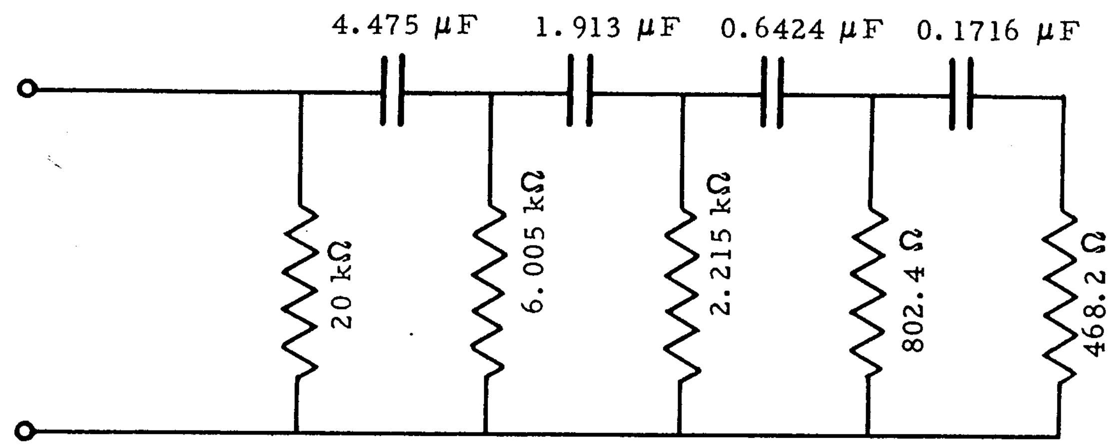

Figure 3. Realization of impedance $z_1(\omega)$, eq (19), with $R_0 = 20\ k\Omega$ and $\tau_2^{(1)} = 0.04$ seconds, which approximates fractional capacitor of order one-half over frequency range (1 Hz, 4 kHz).

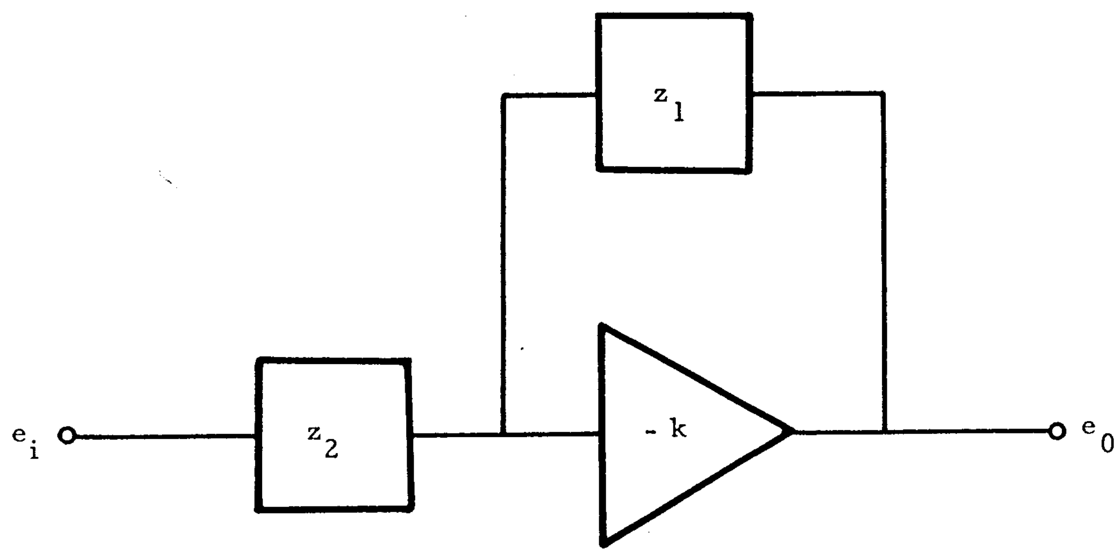

Figure 4. Operational amplifier system with transfer function given by eq (20).

Thus, if one chooses z_2 to be a pure resistance of value R_0, the over-all transfer function is just $G(\omega)$ given by eq (16). Such a device has, in fact, been built and tested at the National Bureau of Standards.

It is interesting to note that if one interchanges the impedances z_1 and z_2 in figure 4, one obtains a prewhitening filter for flicker noise in the frequency range from 1Hz to 4 kHz.

3. EFFICIENT GENERATION OF APPROXIMATE FLICKER NOISE NUMBERS

From the impulse response function

$$h(t) = \frac{1}{\sqrt{t}} \tag{22}$$

corresponding to the transfer function

$$\frac{1}{\sqrt{j\omega}},$$

one can generate a flicker signal $e_0(t)$ from a white noise source $e_i(t)$ by the convolution integral [18]:

$$e_0(t) = \int_0^t d\tau \frac{e_i(\tau)}{\sqrt{t-\tau}} . \tag{23}$$

Discrete flicker numbers can be approximated by a discrete analog convolution, setting $t_n = n\,\delta t$:

$$(e_0)_m = \sum_{n=1}^{m} h(t_m)(e_i)_{m-n}, \tag{24}$$

where the $(e_i)_n$ are uncorrelated random deviates. To obtain a flicker spectral density over a relative frequency range r, the memory length M must be very large and so also the number of remembered deviates $(e_i)_n$ for each output $(e_0)_n$. This is not a very efficient use of computer memory space nor of computer time.

A very efficient flicker number generator can be constructed from the filter cascade introduced in section 2. The differential equation for the network of figure 1 is:

$$y(t) + \dot{y}(t)(R_1 + R_2)C = R_2C\dot{x}(t) + x(t). \tag{25}$$

This can be approximated by the difference equation for the discrete values $y_n = y(n\ \delta t)$:

$$y_{n+1} = \left(1 - \frac{\delta t}{\tau_1 + \tau_2}\right) y_n + \frac{\tau_2}{\tau_1 + \tau_2} x_{n+1} + \frac{\delta t - \tau_2}{\tau_1 + \tau_2} x_n, \tag{26}$$

with, as before, $\tau_j = R_jC$. Let

$$\gamma = \frac{\delta t}{\tau_1 + \tau_2}, \tag{27}$$

and

$$R = \frac{\tau_2}{\tau_1 + \tau_2}. \tag{28}$$

The recursion relation for y_n is then

$$y_{n+1} = (1 - \gamma)y_n + Rx_{n+1} - (R - \gamma)x_n, \tag{29}$$

for an unloaded filter. As in section 2, we shall again consider four such filter sections, with parameters $\gamma^{(i)}$ and $R^{(i)}$, $i = 1, 2, 3, 4$, inputs $x_n^{(i)}$ and outputs $y_n^{(i)}$. Cascading the filters, the output of filter i is the input of filter $i + 1$:

$$\left.\begin{aligned} y_n^{(1)} &= x_n^{(2)} \\ y_n^{(2)} &= x_n^{(3)} \\ y_n^{(3)} &= x_n^{(4)} \end{aligned}\right\}. \tag{30}$$

Note that this is equivalent to assuming the existence of isolation amplifiers between sections of the analog filter. That is, eq (29) gives the exact filter section response (no loading effects) and its output is exactly the input to the next stage (eq (30)).

Then, with the earlier definitions:

$$\gamma^{(i+1)} = \frac{\gamma^{(i)}}{\beta} , \tag{31}$$

$$R^{(i)} = R = 1/3 , \tag{32}$$

and

$$\beta = \left(\frac{\tau_2^{(i)}}{\tau_1^{(i)} + \tau_2^{(i)}} \right)^2 = (1/3)^2. \tag{33}$$

In this situation the higher-numbered filters have the shorter time constants. It seems reasonable to let the shortest time constant, $\tau_2^{(4)}$, be equal to the inverse of the Nyquist frequency, $f_{NQ} = \frac{1}{(2)(\delta t)}$.

Then

$$\delta t = \frac{1}{2} \tau_2^{(4)} \tag{34}$$

and

$$\gamma^{(i)} = \frac{1}{2} \left(\frac{1}{3}\right)^{9-2i} \quad (i = 1, 4) \; . \tag{35}$$

The resulting recursion relations for the four-filter cascade become:

$$\left.\begin{aligned}
x_{n+1}^{(2)} &= y_{n+1}^{(1)} = \left(\frac{4373}{4374}\right) y_n^{(1)} + \frac{1}{3} x_{n+1}^{(1)} - \left(\frac{1457}{4374}\right) x_n^{(1)} \\
x_{n+1}^{(3)} &= y_{n+1}^{(2)} = \left(\frac{485}{486}\right) y_n^{(2)} + \frac{1}{3} x_{n+1}^{(2)} - \left(\frac{161}{486}\right) x_n^{(2)} \\
x_{n+1}^{(4)} &= y_{n+1}^{(3)} = \left(\frac{53}{54}\right) y_n^{(3)} + \frac{1}{3} x_{n+1}^{(3)} - \left(\frac{17}{54}\right) x_n^{(3)} \\
y_{n+1}^{(4)} &= \left(\frac{5}{6}\right) y_n^{(4)} + \frac{1}{3} x_{n+1}^{(4)} - \frac{1}{6} x_n^{(4)} .
\end{aligned}\right\} \tag{36}$$

When the $x_n^{(1)}$ are uncorrelated random deviates, and the $\left\{x_o^{(i)},\ y_o^{(i)}\right\}$ are taken to be zero initially (to minimize the transient), the above recursion equations yield a set of numbers, $y_n^{(4)}$, which approximate flicker noise over a range of about 1000 periods of δt. (The addition of a fifth, lower frequency, or i = 0, stage should extend this range to nearly 10, 000 periods.) The recursion is neatly accomplished on a digital computer with a random number generator and storage locations for the eight distinct coefficients plus the seven numbers $x_n^{(2)}$, $x_n^{(3)}$, $x_n^{(4)}$, $y_n^{(1)}$, $y_n^{(2)}$, $y_n^{(3)}$, $y_n^{(4)}$.

Figure 5 shows a Fortran program written for the generation of 1024 numbers which approximate a flicker noise sample using eqs (36). Figure 6 shows a sample of N = 1024 numbers obtained using this program. This block of data was subjected to a time-domain statistical analysis developed by Allan [19], [20]. Figure 7 shows the Allan variance, $\sigma(\tau)$, for independent samples. For pure flicker noise of arbitrary length, $\sigma(\tau)$ has no τ-dependence. The deviations of the plot from a straight horizontal are probably due to the small sample size.

Figure 8 is a plot of the actual impulse response function of the digital filter described in the Fortran program of figure 5.

4. ACKNOWLEDGMENT

The authors wish to thank Mr. D. W. Allan of the National Bureau of Standards for his assistance in the statistical analysis of the data.

```
      PROGRAM FLICKER
      DIMENSION V(5, 2)
C     SET INITIAL VALUES V(I, 1) TO ZERO MEAN VALUES
      DO 1  I=1, 5
    1 V(I, 1) = 0.
C     GENERATE AND PRINT N FLICKER NUMBERS
      N = 1024
      DO 2 I=1, N
C     SELECT RANDOM V(1, 2) UNIFORMLY DISTRIBUTED ON (-½, ½)
      V(1, 2) = RANF (-1) -0.5
C     SOLVE RECURSION RELATIONS
      V(2, 2) = .999771*V(2, 1) + .333333*V(1, 2) - .333105*V(1, 1)
      V(3, 2) = .997942*V(3, 1) + .333333*V(2, 2) - .331276*V(2, 1)
      V(4, 2) = .981481*V(4, 1) + .333333*V(3, 2) - .314815*V(3, 1)
      V(5, 2) = .833333*V(5, 1) + .333333*V(4, 2) - .166667*V(4, 1)
      PRINT 11,  V(5, 2)
   11 FORMAT (2XF12.6)
C     RESET V
      DO 3 J=1, 5
    3 V(J, 1)=V(J, 2)
    2 CONTINUE
      CALL EXIT
      END
```

Figure 5. Fortran program for computing N approximate flicker numbers by recursion relations, eq (36). ("N = 1024" has been specified.)

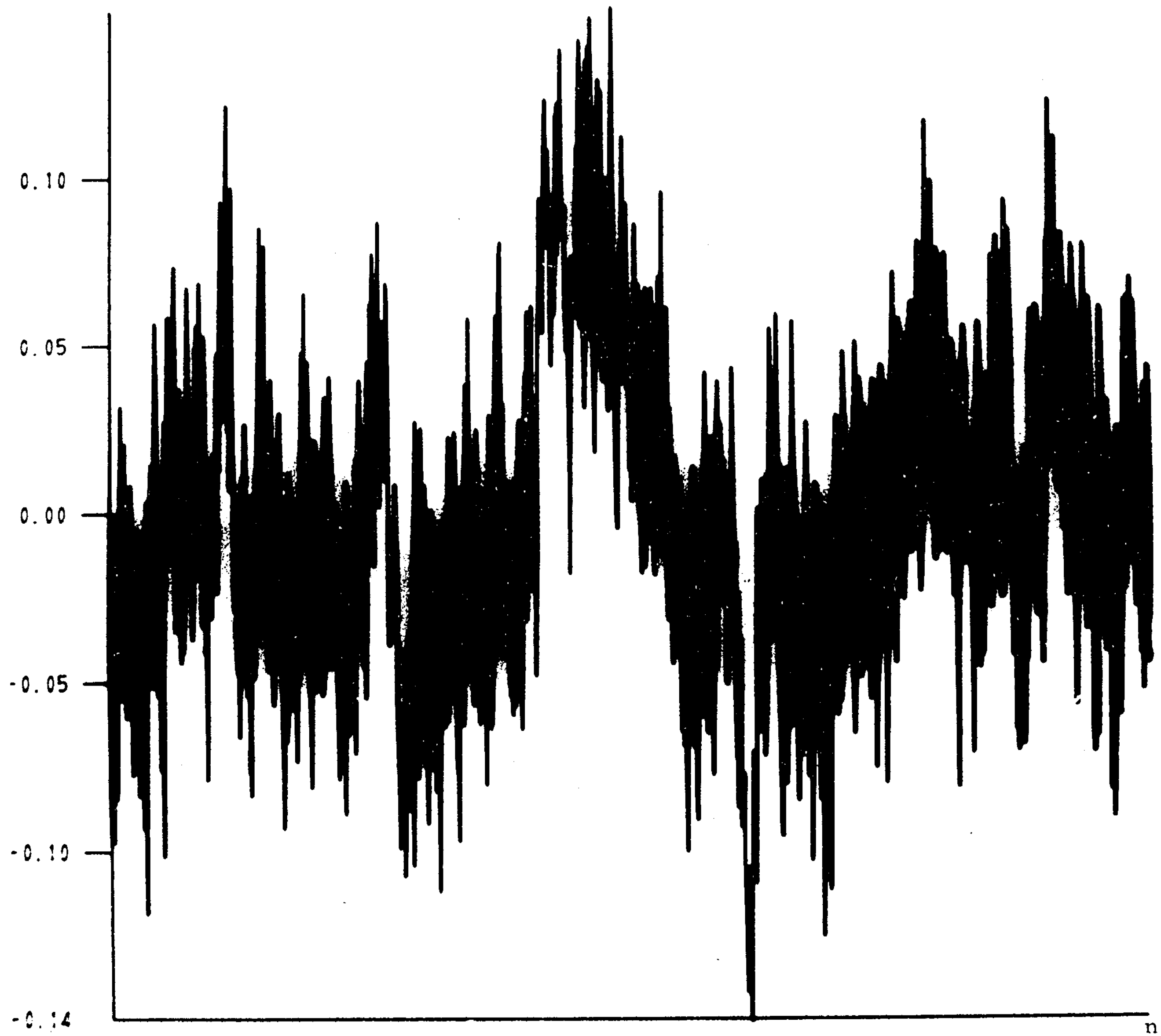

Figure 6. Sample of 1024 approximate flicker numbers generated from program of figure 5.

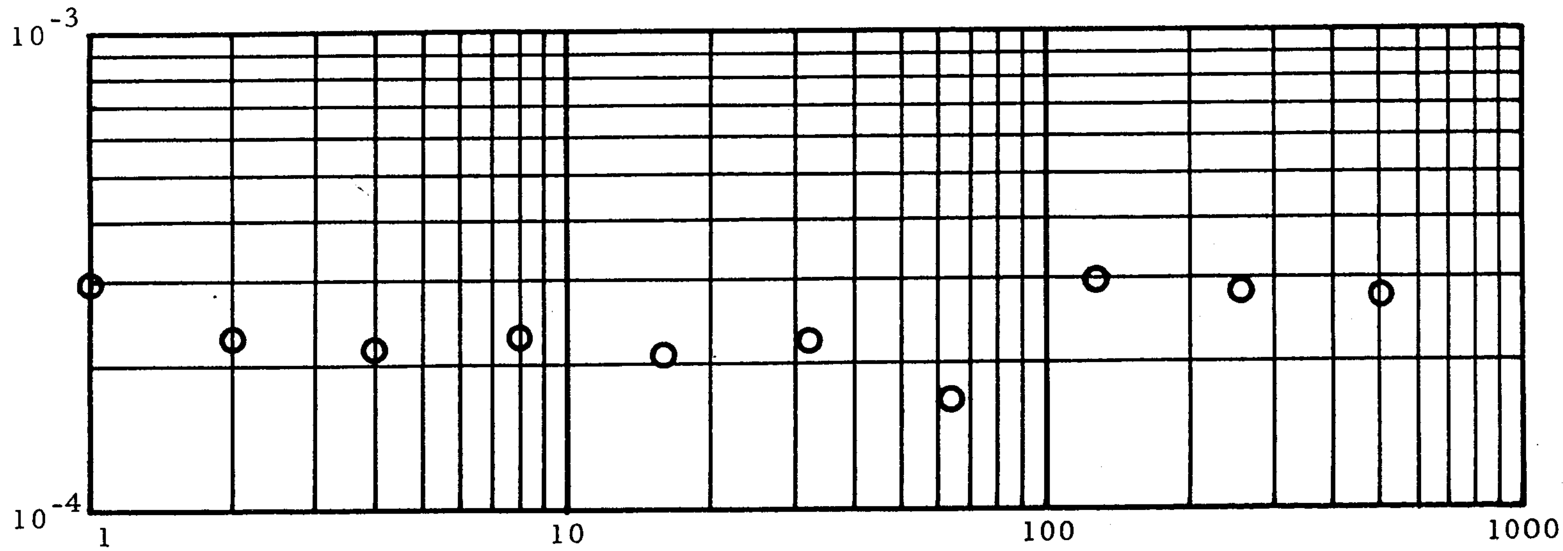

Figure 7. Square root of Allan variance $\sigma(\tau)$ for data sample of figure 6.

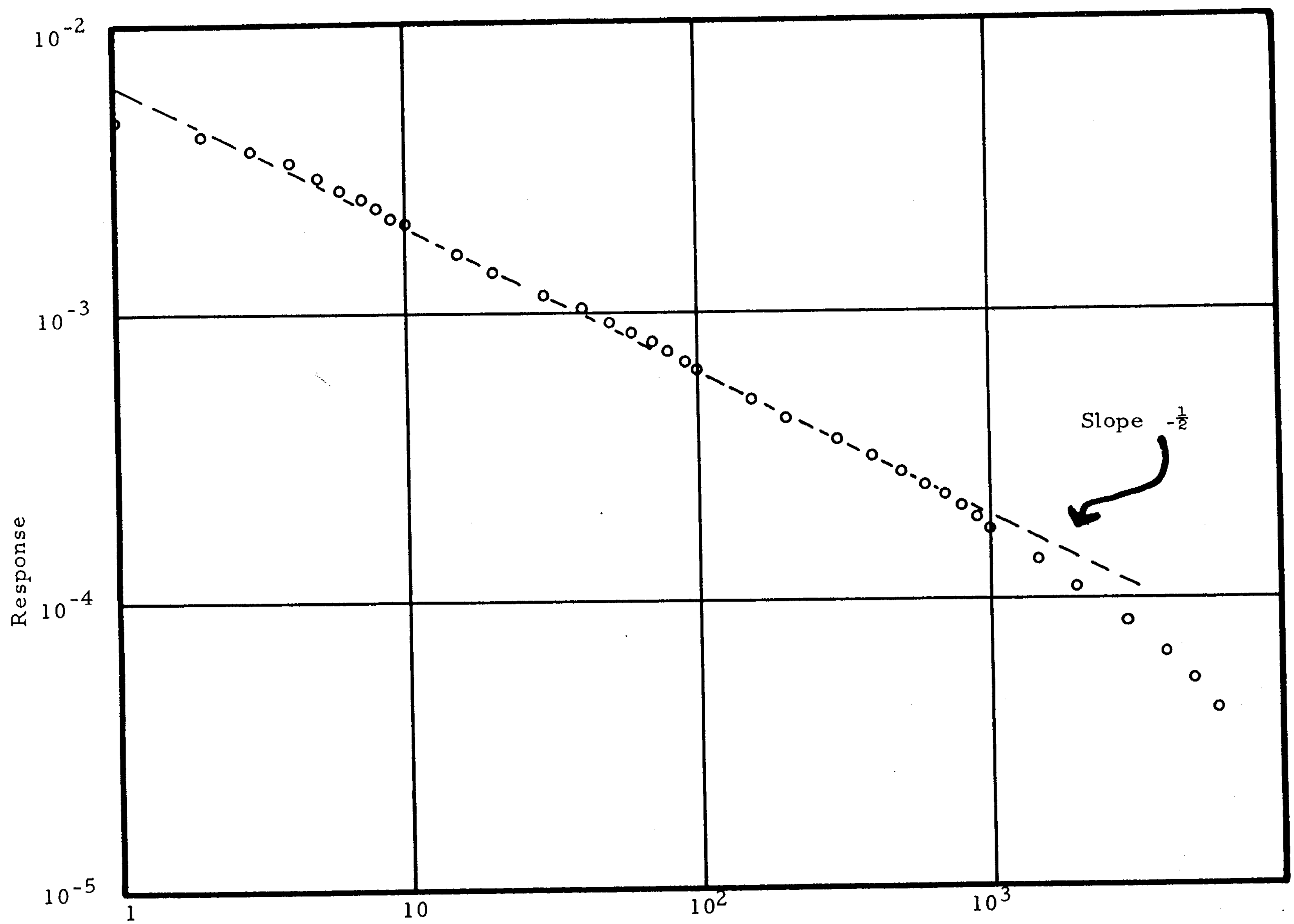

Figure 8. Impulse response function $y_n^{(4)}$ of digital filter.

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