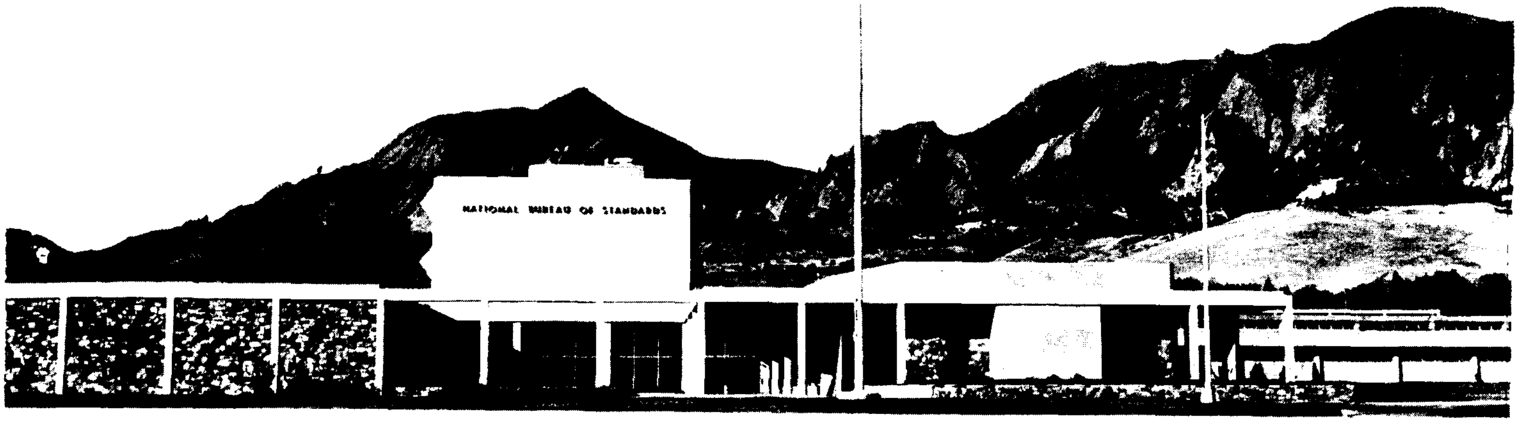

NBS REPORT

9700

ON THE CORRECT-READING RATE AND THE DISTRIBUTION OF TIME BETWEEN CORRECT READINGS FOR CLOCK ENSEMBLES

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NATIONAL BUREAU OF STANDARDS

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TABLE OF CONTENTS

ON THE CORRECT-READING RATE AND THE DISTRIBUTION OF TIME BETWEEN CORRECT READINGS FOR CLOCK ENSEMBLES

G. E. Hudson

Clocks in certain kinds of statistical ensembles can be characterized in terms of the instantaneous fractional rate of those which read correctly, and the probable distribution of times between correct readings or "zero crossings." An integral equation of standard renewal type relates these quantities. If the dispersion of the ensemble varies with time in certain ways, corresponding solutions of this equation can be and are derived, utilizing Laplace transforms. Some are applicable, for example, to trends of the dispersion function which correspond to white noise, random walk noise, and flicker noise fluctuations of the clocks' readings about the ensemble average.

Key Words: correct-reading rate of clocks, flicker noise, random walk noise, renewal processes, statistical clock ensembles, time distribution between correct readings of clocks, time-keeping, white noise.

1. INTRODUCTION

The initial purpose of this paper is to present and to a minor extent discuss a formulation of a mathematical description of the statistical behavior of an ensemble of clocks. A model of this ensemble may be specified in terms of the distribution of its clock readings about the ensemble average. This model is specified in more detail in the next sections, as suggested by certain experimental studies, but it is not the purpose of this paper to discuss or present details of these experiments; rather, it is the purpose simply to deduce properties and behaviors of the model so suggested.

The description will be seen to lead to a certain integral equation (or summation equation, in the discrete case) relating two functions of time whose trends are significant in the study and classification of clock reading statistics.

The main purpose of the paper is to determine and exhibit pairs and classes of solution functions satisfying this equation and study their properties. Cases of particular importance, since they relate to types of clock reading behaviors designated by the terms white noise, random walk noise, and flicker noise, are studied, and lead to explicit formulas describing the behavior of the ensemble for large and small times.

Difficulties of this mode of description associated with the case of flicker noise and, more generally, ensemble dispersions, $\sigma(t)$, with rates which are proportional to powers of time greater than or equal to unity, are noted. A possible direction for generalization of the description is pointed out, which introduces a statistical mechanical representation of the ensemble in a two-dimensional phase space.

The results and methods of this paper should be regarded as preliminary; certain ones of them are new, as are the point-of-view and suggestions for future work. Since the integral equation studied is of standard renewal type, many results of studies of such processes are immediately applicable, but are not reviewed herein.

2. SPECIFICATION OF THE ENSEMBLE TYPE

Let us consider a statistical ensemble of identically constructed independently running clocks contained in a small enough spatial region so that comparisons of their simultaneous readings, τ, can be made, effectively, in an instantaneous fashion. Assume that, at any instant, the average reading, t, of the ensemble exists, as well as its variance, σ^2. We shall, on occasion, for convenience refer to the ensemble average time, t, as the "correct" time.

Thus,

$$t = \langle \tau \rangle \tag{1a}$$

$$\sigma^2 = \langle (\tau - t)^2 \rangle \tag{1b}$$

where the brackets denote the ensemble average. Let the random variable ξ denote the deviation of the ensemble clock readings from the average, so that

$$\xi = \tau - t$$

and hence

$$\langle \xi \rangle = 0. \tag{1c}$$

We shall also suppose in general that σ^2 depends on "the time", t,

$$\sigma = \sigma(t). \tag{2a}$$

It is further specified that, at one particular instant, all the clocks are set to have the same reading $t = 0$, so that, at this instant

$$\sigma(0) = 0. \tag{2b}$$

Moreover, they are adjusted at this initial instant to run at the same nominal rates. Nevertheless, thereafter they diverge on the average from t in their readings because each rate (and reading) is subject to random statistical fluctuations, which produce the hypothesized distributions as well as other statistical characteristics to be described.

Experiments in which the frequency of quartz oscillators were measured repeatedly against a standard, suggest an empirical model (Ref 1). If the oscillator were used to drive a clock, then it has been shown empirically that the readings of an ensemble of such clocks would have the probability density

$$\varphi(\xi, t) = \frac{1}{\sqrt{2\pi}\,\sigma(t)} e^{-\frac{\xi^2}{2\sigma^2(t)}} \tag{3a}$$

which satisfies the foregoing relations, with the additional condition that

$$\varphi(\xi, 0) = \delta(\xi) \quad . \tag{3t}$$

For the stationary, or "white noise" case, σ is a constant.

3. A BASIC RELATIONSHIP

According to J. Barnes (Ref 1) the clock ensemble of the kind considered here can additionally be characterized in terms of two other functions, $\rho(t)$ and $r(t)$, as follows. It may be assumed that the random deviations of the clock readings from their average are time varying in such a way that each clock might read the correct time, again and again, according to some probability law. That is, if a clock reads the value t' at the instant when the ensemble average is t', there is a conditional probability density $p(t, t') \geq 0$ that the clock will again read the "correct" time, t, but not read the correct value at any intervening instant. Moreover, it is assumed that this probability density is stationary, in the sense that p depends only on the difference $t - t'$, i.e.,

$$p(t, t') = \rho(t-t') \, . \tag{4a}$$

Statistically, then, we shall regard

$$\rho(t'-t'')dt' \tag{4b}$$

as the probable number of clocks in the ensemble which do read t' between times t' and $t' + dt'$, on the condition that it read the correct time t'' at an earlier "time" t'', but did not read the correct time between t' and t''.

This conditional probability is related, according to Barnes, to an (unconditional) rate distribution function $r(t)$. The differential

$$r(t')dt' \tag{5}$$

represents the fractional number of ensemble clocks which have the correct reading t' when the ensemble average is between the times t' and $t' + dt'$. We shall call $r(t)$ the "correct-reading" rate of the ensemble.

Because of the hypothesized initial δ-function distribution, it is only after the instant $t' = 0$ that any of the clocks can again read "correctly". Subsequently, the portion of those clocks which did read correctly at time t', and which again show the correct time, t, at a later moment t, but not in between, is

$$r(t')\rho(t-t')dtdt' \quad , \qquad t \geq t'.$$

Now, at time $t \geq 0$, the number of clocks reading correctly in the interval dt, must equal the number

$$\rho(t)dt$$

which read correctly at time, t, having also read zero at $t = 0$ (but not in between) <u>plus</u> the number which did read correctly at times t' between 0 and t but which did not read correctly between t' and t, i.e.,

$$\int_0^t r(t')\rho(t-t')dt'dt.$$

Thus, we arrive at the integral equation relating $r(t)$ and $\rho(t)$:

$$r(t) = \rho(t) + \int_0^t r(t')\rho(t-t')dt' \quad . \tag{6}$$

This is Barnes' result. It has the form of a standard "renewal equation", as in the theory of probability (Ref 2).

This integral equation shows that r and ρ can be regarded as so-called "reciprocal kernels", as in the study of linear integral equations of second kind, of Volterra type. In fact, under certain conditions, if

$$F(t) = G(t) + \int_0^t \rho(t-t')F(t')dt'$$

is an equation for F, given G and ρ, then

$$F(t) = G(t) + \int_0^t r(t-t')G(t')dt'$$

provided ρ and r are related by the integral equation derived in the foregoing (Ref 3).

The specification of the probability density $\rho(t)$ is not yet complete. It is reasonable to require that

$$\rho(t) \begin{cases} \geq 0 \ , & t \geq 0 \qquad (7a) \\ = 0 \ , & t < 0 \ . \qquad (7b) \end{cases}$$

If one were certain that no clock remained permanently incorrect in its reading, then one could postulate that

$$\int_0^\infty \rho(t)dt = 1. \qquad (8)$$

Equation (7b) states that the probable number of clocks which did read the time correctly in a small time interval at an earlier time, as a result of the fact that they sometime later will read the correct time is zero. Interpreted in causal terms, Equation (7b) like Equation (8) would imply a restriction on the physical nature of the clocks; but it can also be regarded as a convenience in completing the specification of $\rho(t)$ so that some solutions of Equation (6) might be obtained more easily using it. Equation (8), too, is posited in the remainder of this paper, and should be regarded as a convenient normalization of ρ. More generally $\int_0^\infty \rho dt \leq 1$; for, as was noted by W. Feller (Ref 4), ρ is a conditional probability; or it might be a rate, like r, as Barnes too has suggested (Ref 1). In any case (8) need not hold in general.

It is possible to set up a difference equation analogous to Equation (6), which has sometimes been useful in making numerical computations.

This is

$$r_n = \rho_n + \sum_{n'=1}^{n-1} \rho_{n-n'} r_{n'} , \qquad (n>0) \tag{9}$$

and in analogy to Equation (7) and (8),

$$\rho_n \begin{cases} \geq 0 , & n>0 & (10a) \\ = 0 , & n\leq 0 & (10b) \end{cases}$$

and

$$\sum_{n=1}^{\infty} \rho_n = 1 . \tag{11}$$

The derivation of (9) is as follows: Consider again an ensemble of clocks, and let there be a series of readings of these clocks at discrete instants t_n, $(n=1, 2, \cdots)$, such that the ensemble average of the deviations, ξ_n, of the readings from the average, t_n, is zero. Assume further that at t_n there is a probability, $\rho_{n-n'}$, that a clock of the ensemble shall have a deviation between $-\Delta t/2$ and $+\Delta t/2$ from the average reading, under the condition that at time $t_{n'}$ its readings also lay in this range, but did not do so for any reading between $t_{n'}$ and $t_n (n'<n)$. Suppose that ρ depends only on $(n-n')$ (stationarity). Let r_n be the fractional number of clocks at time t_n which have deviations in the range $(-\Delta t/2, +\Delta t/2)$. Then, $r_{n'}\rho_{n-n'}$ is the fractional number of clocks at time t_n whose readings lie between $t_n - (\Delta t/2)$ and $t_n + (\Delta t/2)$ and which also lie in this range at time $t_{n'} (n'<n)$, but not at any instant between. Equation (9) follows immediately; (10a) is required since ρ_n is a probability, and the stipulations (10b) and (11) may be imposed as a convenience in the solution of (9) or as part of the definition of the ensemble and the clock mechanism. An equation like (9) can also be derived from (6), by introducing power series expansions of $r(t)$ and $\rho(t)$.

Note that no <u>physical</u> reason for the clocks behaving in the supposed way as described (either by (9) or (6)) has been given. Indeed, both the integral and difference equations are valid by definition of the quantities appearing in them. If the definitions are self-consistent and applicable to reality, there are necessarily pairs of solutions r and ρ, with the stated properties, which we shall consider eventually. Whether or not there exist physically reasonable ensembles of these types is a different question.

Still another problem to which we now turn is the relation between $\sigma(t)$, or the probability density $\varphi(\xi, t)$, and the correct reading rate distribution $r(t)$.

4. THE JOINT READING AND RATE DISTRIBUTION

As before, consider an ensemble of clocks with a random variable clock-reading, τ. However, let us now introduce the notion of clock-rate, $\dot{\tau}$, as a random variable. There are some indications, as we shall point out, that this generalization will probably be distinctly advantageous in more extensive developments of the theory than can be dealt with here. If a clock reading has a rate $\dot{\tau}$, different from unity, it must be derived with respect to some other suitable time dependent quantity. We shall choose the ensemble average as this quantity, so that

$$\dot{\tau} = \frac{d\tau}{dt} \tag{12a}$$

for any given clock of the ensemble. Sometimes one also speaks of the clock-rate offset (from the average)

$$\epsilon = \frac{d\tau}{dt} - 1. \tag{12b}$$

We shall assume that the ensemble is characterized by a two-dimensional density function, $D_t(\tau, \dot{\tau})$, defined over the "phase-space" of the independent random variables τ and $\dot{\tau}$.

In the general theory, it might be supposed in addition that a typical clock of the ensemble, except for random perturbations to which it is subjected, can be specified in terms of some Hamiltonian (generally with a very large number of variables, and possibly having a quantum-physical basis). Many further results, following upon the use of

Liouville's theorem and other statistical mechanical considerations, await further development. However, a few relationships can be found, based simply on the concepts and definitions introduced up to this point.

From its definition, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_t \, d\tau \, d\dot{\tau} = 1 \quad . \tag{13}$$

Also, the time indicated by the ensemble is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau D_t \, d\tau \, d\dot{\tau} = t \tag{14}$$

whereas, the average ensemble rate is unity

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dot{\tau} D_t \, d\tau \, d\dot{\tau} = 1. \tag{15}$$

The relation between the ensemble probability density, $\varphi(\tau - t, t)$, over the clock-reading τ, and the phase distribution D_t is readily seen to be

$$\int_{-\infty}^{\infty} D_t(\tau, \dot{\tau}) d\dot{\tau} = \varphi(\tau - t, t) \quad . \tag{16}$$

This must hold, whether or not the density φ is the Gaussian one mentioned earlier. It is interesting, as a trivial check, to calculate the fractional number of clocks having a "correct" reading between t and t+dt, at the ensemble time t. Clearly, this is

$$dt \int_{-\infty}^{\infty} D_t(t, \dot{\tau}) d\dot{\tau} = r(t) dt \quad , \tag{17}$$

so that we have established formally the general connection, intuitively obvious (Ref 1),

$$\varphi(0, t) = r(t) \tag{18}$$

between φ and the rate distribution function, r, for zero deviations.

In the special case of a Gaussian density, Equation (3a), with dispersion $\sigma(t)$, we have also derived the important result

$$r(t) = \frac{1}{\sqrt{2\pi}\,\sigma(t)} \quad . \tag{19}$$

Finally, if desirable, the initial density-in-phase-space can be specified to have the distribution

$$D_o(\tau, \dot{\tau}) = \delta(\tau)\delta(\dot{\tau} - 1) \quad , \tag{20a}$$

or more generally

$$D_o(\tau, \dot{\tau}) = \delta(\tau)u(\dot{\tau}) \tag{20b}$$

where

$$\int_o^{\infty} u(\dot{\tau})d\dot{\tau} = 1$$

and

$$\int_o^{\infty} \dot{\tau}u(\dot{\tau})d\dot{\tau} = 1.$$

Additional and more specific information concerning the behavior of D_t, φ, ρ, and r, other than that contained in or implied by the foregoing equations, must await more explicit specification of the physical and statistical properties of clocks.

5. SOME SOLUTIONS AND PROPERTIES OF SOLUTIONS

We now turn to a consideration of the implications of the integral or summation equation (6) or (9), taken in conjunction with conditions (7) and (8), or (10) and (11). First, let us deal with the integral equation.

5.1. The Continuous Case

Integration of Equation (6) from 0 to ∞, with an interchange of the order of integration on t' and t, and using (7) and (8), leads easily, as in Appendix I, to the result,

$$\int_0^\infty r(t)dt = \infty \quad . \tag{21}$$

Let us assume that the one-sided Laplace-transform, $\tilde{r}(\lambda)$, of r(t) exists:

$$\tilde{r}(\lambda) = \int_0^\infty e^{-\lambda t} r(t)dt. \tag{22}$$

Condition (8) insures the existence of $\tilde{\rho}(\lambda)$ for $\rho(t)$. Then, as shown in Appendix II, an application of (7) and (8) with an integration procedure similar to that leading to (21), shows that the Laplace transform of the "convolution" integral in (6) is simply the product of $\tilde{r}$ and $\tilde{\rho}$. The Laplace transform of (6) thus leads to

$$\tilde{r} = \frac{\tilde{\rho}}{1-\tilde{\rho}} \tag{23a}$$

and

$$\tilde{\rho} = \frac{\tilde{r}}{1+\tilde{r}} \quad . \tag{23b}$$

It is easily checked that since

$$\tilde{\rho}(0) = \int_0^\infty \rho(t)dt = 1 \tag{24a}$$

then $$\tilde{r}(0) = \infty \tag{24b}$$

as it should be.

Set $$\tilde{\rho} = 1 - \frac{\lambda^{\beta}}{f(\lambda)}, \quad f(0) = \mu \tag{25a}$$

with $\beta > 0$. Then

$$\tilde{r} = \frac{f(\lambda)}{\lambda^{\beta}} - 1 \ . \tag{25b}$$

As $\lambda \to \infty$, most decent Laplace transforms approach 0 (Ref 4). Hence

$$f(\lambda) = O\left(|\lambda^{\beta}|\right), \text{ as } \lambda \to \infty \tag{26}$$

With these stipulations, we may reasonably expect, given $\rho(t)$, that $\tilde{\rho}$ and hence $\tilde{r}$ can be found, and then that the inverse transform of $\tilde{r}$ would lead to $r(t)$. Or if $r(t)$ is given, the inverse Laplace transformation (Ref 5)

$$\rho(t) = \frac{1}{2\pi i} \lim_{\delta \to \infty} \int_{\gamma - i\delta}^{\gamma + i\delta} e^{zt} \frac{\tilde{r}(z)}{1 + \tilde{r}(z)} dz \tag{27}$$

(as in Appendix III) or some equivalent, more convenient, procedure would yield the conditional probability density. Note that the special case, to which we return presently, for which

$$\left.\begin{aligned} f(\lambda) &= \lambda^{\beta} + \mu \\ \text{and} \qquad \tilde{r} = \mu/\lambda^{\beta}, \ \tilde{\rho} &= \mu\Big/\left(\mu + \lambda^{\beta}\right) \end{aligned}\right\} , \ \beta > 0 \tag{28}$$

satisfies all the foregoing conditions. Notice also that the simple but important special case

$$\tilde{\rho} = \frac{P_n(\lambda)}{Q_m(\lambda)} \tag{29a}$$

where $m > n$, and P_n and Q_m are polynomials of degree n and m respectively such that

$$P_n(0) = Q_m(0) \neq 0, \tag{29b}$$

leads to

$$\tilde{r} = \frac{P_n(\lambda)}{Q_m(\lambda) - P_n(\lambda)} \tag{29c}$$

which is a similar ratio of polynomials. Generalization of (29) is, of course, easily accomplished. Let us now consider a variety of cases which are particularizations of the foregoing.

a. Exponential Distributions

Choose r(t) to have the form

$$r(t) = \sum_{n=1}^{N} \bar{r}_n e^{-\alpha_n t}, \quad \alpha_n \geq 0, \tag{30}$$

so that, since

$$\tilde{r}(\lambda) = \sum_{n=1}^{N} \frac{\bar{r}_n}{\lambda + \alpha_n} \quad , \tag{31}$$

this is an example of Equation (29a-c).

To satisfy the condition $\tilde{r}(0) = \infty$, let

$$\alpha_1 = 0 . \tag{31a}$$

Define

$$Q_N(\lambda) = \lambda \prod_{n=2}^{N} (\lambda + \alpha_n) \tag{32a}$$

and

$$P_{N-1}^{(n)}(\lambda) = \prod_{i \neq 1}^{N} \frac{\lambda + \alpha_i}{\lambda + \alpha_n} \quad . \tag{32b}$$

Then

$$\tilde{\rho}(\lambda) = \frac{\sum_{n=1}^{N} \bar{r}_n P_{N-1}^{(n)}(\lambda)}{Q_n(\lambda) + \sum_{n=1}^{N} \bar{r}_n P_{N-1}^{(n)}(\lambda)} \tag{33a}$$

$$= \sum_{n=1}^{N} \frac{\bar{\rho}_n}{\lambda + \beta_n} \tag{33b}$$

where $\bar{\rho}_n$ are the numerators in the expression of $\tilde{\rho}$ in partial fractions, and where the roots of the Nth order polynomial in the denominator of $\tilde{\rho}(\lambda)$ are denoted by $-\beta_n$ (and are assumed to be distinct). Because $Q_N(0) = 0$, we also have

$$\tilde{\rho}(0) = \sum_{n=1}^{N} \frac{\bar{\rho}_n}{\beta_n} = 1 \tag{34a}$$

as well as, for $m > 1$,

$$\tilde{\rho}(-\alpha_m) = \sum_{n=1}^{N} \frac{\bar{\rho}_n}{\beta_n - \alpha_m} = 1 \quad . \tag{34b}$$

Define

$$R_N(\lambda) = Q_N(\lambda) + \sum_{n=1}^{N} \bar{r}_n P_{N-1}^{(n)}(\lambda) = \prod_{i=1}^{N} (\lambda + \beta_i) \tag{35a}$$

so that

$$R_N(-\beta_n) = 0 \quad ; \tag{35b}$$

also, let

$$S_{N-1}^{(n)}(\lambda) = \prod_{i=1}^{N} \frac{\lambda + \beta_i}{\lambda + \beta_n} \quad . \tag{35c}$$

Then we see from Equations (33) and (35), after setting $\lambda = -\beta_n$, that

$$\bar{\rho}_m = \frac{\sum_{n=1}^{N} \bar{r}_n P_{N-1}^{(n)}(-\beta_m)}{S_{N-1}^{(m)}(-\beta_m)} \quad , \ (m = 1, \cdots, N) \ . \tag{36}$$

The inverse transform for distinct roots, $-\beta_n$, yields

$$\rho(t) = \sum_{n=1}^{N} \bar{\rho}_n e^{-\beta_n t} \quad . \tag{37}$$

An important case of indistinct roots is considered presently.

For N = 1,

$$r = \bar{r}_1 \tag{38a}$$

and

$$\rho = \bar{r}_1 e^{-\bar{r}_1 t} \tag{38b}$$

which is a check on the reasonableness of the theory. To summarize, if $r(t)$ is a sum of exponentials, plus a constant, then $\rho(t)$ is a sum of exponentials. In particular, if the "correct reading rate" has a constant value $\bar{r}_1$, the probability density that a clock will return to the correct reading for the first time, after a period of time, t, decreases with t from the value $\bar{r}_1$, at an exponential decay rate equal to $\bar{r}_1$. In this situation, if the ensemble has a Gaussian probability density (Equation (3a)) in the deviations of its clock readings from the ensemble average, then the dispersion, σ, is a constant, according to Equation (19).

$$\sigma = \frac{1}{\sqrt{2\pi}\,\bar{r}_1} \tag{39}$$

as in "white noise".

Let us assume, as another important special case, with k being an integer, that

$$\rho(t) = \frac{\mu^k t^{k-1}}{\Gamma(k)} e^{-\mu t} \tag{40}$$

which is a Poisson density distribution satisfying

$$\int_0^\infty \rho(t)dt = 1 \quad , \quad (k > 0) .$$

Then

$$\tilde{\rho}(\lambda) = \left(\frac{\mu}{\lambda + \mu}\right)^k , \tag{41}$$

and we see that the pole of $\tilde{\rho}$ is of order k, a case of non-distinct roots.

Accordingly

$$\tilde{r}(\lambda) = \frac{1}{\left(\frac{\lambda}{\mu} + 1\right)^k - 1} \quad . \tag{42}$$

For integral k-values, the denominator vanishes for

$$\frac{\lambda}{\mu} = \left(\omega_o - 1,\ \omega_1 - 1,\ \cdots,\ \omega_{k-1} - 1\right) \tag{43}$$

where ω_ℓ ($\ell = 0, \cdots k-1$) are the complex kth roots of unity,

$$\omega_\ell = e^{2\pi i \frac{\ell}{k}} \quad . \tag{44}$$

Hence

$$r(t) = \mu \sum_{s=0}^{k-1} \left(\frac{e^{\mu(\omega_s - 1)t}}{\prod_{\ell=0}^{k-1}{}' \ (\omega_s - \omega_\ell)} \right) \tag{45}$$

where the prime indicates omission of the factor for which $\ell = s$. Because of the nature of the roots of unity, $r(t)$ is real, but has damped oscillatory terms for $k > 2$. There may be a k-value for which r is negative for some t. This is an open question. It is clear that the case $k = 1$ is identical with the former "white noise" case.

b. Power Laws

We now turn to a situation of great importance (Ref 6), for which $r(t)$ is proportional to some power, $\beta - 1$, of t. Now cases for which β is greater than 1 are unusual for independent clocks, in view of the inverse relation between r and σ. Cases have been considered (Ref (9)) for which the ensemble dispersion decreased with time! The case for which $\beta = 1$, has already been discussed; this is the "white noise" case. A value $\frac{1}{2}$ for β corresponds to the "random walk" (the integral of a white noise) case (Ref 7), while β equals 0 for the integral of a "flicker noise", insofar as the behavior of σ is concerned. Values of β less than zero are certainly interesting and should be studied. However, a grave difficulty arises even for $\beta = 0$. The basic integral equation (Equation (6)) becomes unrealistic for this case; for small enough t, quite generally $r(t)$ and $\rho(t)$ approach equality (provided the convolution integral makes sense). But for $\beta \le 0$, the convolution integral does not converge, near $\tau = 0$; moreover, the integral of $\rho(t)$ diverges because of this behavior of $r(t)$. Both considerations indicate that a fundamental change in this theory is required for $\beta \le 0$. For these reasons, we restrict most of the present discussion to that for $1 \ge \beta > 0$, together with some remarks on the limiting situation, $\beta = 0$.

Choose

$$r(t) = \mu \frac{t^{\beta-1}}{\Gamma(\beta)} = \beta\mu \frac{t^{\beta-1}}{\Gamma(1+\beta)} . \qquad (46)$$

Then $\tilde{r}(\lambda) = \mu/\lambda^{\beta}$, so that we are returning to the situation specified in Equation (28). The general inversion integral, Equation (27), can be evaluated to find $\rho(t)$. This leads, as shown in Appendix III, to the representation of $\rho(t)$ in various integral forms

$$
\rho(t) = \begin{cases} \dfrac{\sin \pi\beta}{\pi \mu t^{1+\beta}} \displaystyle\int_0^{\infty} e^{-\nu} \frac{\nu^{\beta} d\nu}{1 + \dfrac{2\cos\pi\beta}{\mu}\left(\dfrac{\nu}{t}\right)^{\beta} + \dfrac{1}{\mu^2}\left(\dfrac{\nu}{t}\right)^{2\beta}} & (47a) \\ \text{useful for large } t \text{ (i.e., } t \to \infty\text{), and } \beta \neq 1 \text{, or} & \\ \dfrac{\mu t^{\beta - 1}}{2\pi i} \displaystyle\int_0^{\infty} e^{-\nu} \frac{d\nu}{\nu^{\beta}} \left[\frac{e^{i\pi\beta}}{1 + \mu\left(\dfrac{t}{\nu}\right)^{\beta} e^{i\pi\beta}} - \frac{e^{-i\pi\beta}}{1 + \mu\left(\dfrac{t}{\nu}\right)^{\beta} e^{-i\pi\beta}} \right] & (47b) \\ \text{useful for small } t \text{ (i.e., } t \to o\text{), or} & \\ \mu e^{-\mu t} \quad \text{for } \beta = 1. & (47c) \end{cases}
$$

It is helpful to discuss a few limiting situations.

First, consider the asymptotic form for very large t-values. Using Equation (47a) and expanding the integrand in powers of $(\nu/t)^{\beta}$, followed by integration with respect to ν (justifiable by applying a suitable form of Watson's Lemma (Ref 8)), we find, for large t,

$$
\rho(t) \simeq \frac{\sin \pi\beta}{\pi t} \left\{ \frac{\Gamma(\beta+1)}{\mu t^{\beta}} - 2\cos\pi\beta \, \frac{\Gamma(2\beta+1)}{(\mu t^{\beta})^2} + O\left[(\mu t^{\beta})^{-3}\right] \right\} \tag{48}
$$

which even yields $\rho(t) \simeq 0$ for $\beta = 1$ (and large t, of course).

This asymptotic result is of considerable interest, in particular for $\beta = 1/2$, which corresponds to r(t) varying as $t^{-1/2}$, so that the standard deviation of clock readings, Equation (19), varies as $t^{1/2}$; we see that $\rho(t)$ varies as $t^{-3/2}$ in agreement with Chandrasekhar (Ref 7). This is the case known as "random walk", or "Brownian motion." Other cases are also of interest; currently, the case in which $\beta \to 0$, known as "flicker noise" of rate is receiving a great deal of attention (Ref 6). As mentioned before, the case $\beta = 1$ corresponds to "white noise."

For large t-values, the ratio ρ/r has the asymptotic behavior

$$\frac{\rho(t)}{r(t)} \simeq \frac{\Gamma(1+\beta)}{\Gamma(1-\beta)} \frac{1}{(\mu t^{\beta})^2} + O\left[(\mu t^{\beta})^{-3}\right] \tag{49a}$$

for $\beta \neq 1$; explicitly we have

$$\frac{\rho(t)}{r(t)} = e^{-\mu t} \tag{49b}$$

for $\beta = 1$.

Next, consider the behavior of $\rho(t)$ for small values of t. From Equation (47b) we find, after expansion of the integrands in powers of $(t/\nu)^{\beta}$ and integration with respect to ν, that

$$\rho(t) = \frac{\beta}{t} \sum_{n=0}^{\infty} (n+1)(-)^n \frac{(\mu t^{\beta})^{n+1}}{\Gamma[1+\beta(n+1)]} \quad . \tag{50}$$

This converges even for $\beta = 1$ to the correct value, and shows the behavior near $\beta = 0$. The trend of $\rho(t)$ near $t = 0$ for $\beta = 1/2$ is as $t^{-1/2}$ just as for r(t). Indeed the ratio (ρ/r) has the interesting behavior:

$$\lim_{t\to 0} \frac{\rho(t)}{r(t)} = \frac{\beta\Gamma(\beta)}{\Gamma(1+\beta)} = 1 \quad . \tag{51}$$

This is not so for large t, and should only be applied for $\beta > 0$, as t approaches zero.

It may be worthwhile to summarize these limiting properties of r and ρ for $\beta = 1$, $1/2$, and β near 0.

i) White Noise: $\beta = 1$

$$\left.\begin{array}{l} r(t) = \mu \\ \rho(t) = \mu e^{-\mu t} \end{array}\right\} \text{, for all } t \quad . \tag{52a}$$

ii) Random Walk: $\beta = 1/2$

$$r(t) = \frac{\mu}{\sqrt{\Pi}\, t^{1/2}}$$

$$\rho(t) \cong \begin{cases} \dfrac{1}{2\mu\sqrt{\Pi}\, t^{3/2}} & \text{, large t, i.e., as } t \to \infty \\[2ex] \dfrac{\mu}{\sqrt{\Pi}\, t^{1/2}} & \text{, small t, i.e., as } t \to 0. \end{cases} \tag{52b}$$

iii) Flicker rate: $\beta \sim 0$

For this case, let us retain factors of β (which might be incorporated in a renormalization of μ), and of $t^{\pm\beta}$.

$$r(t) \cong \beta\mu t^{\beta-1}$$

$$\rho(t) \cong \begin{cases} \dfrac{\beta}{\mu t^{1+\beta}} & \text{, large t} \\[2ex] \dfrac{\beta\mu t^{\beta-1}}{(1+\mu t^{\beta})^2} & \text{, small t} \quad . \end{cases} \tag{52c}$$

Note that the function describing the behavior of ρ for small t and small β also gives the correct asymptotic behavior for large t and small β -- hence it may be a close approximation for all t. This statement is strengthened by the observation that

$$\int_0^\infty \frac{\beta\mu t^{\beta-1} dt}{(1+\mu t^\beta)^2} = 1 \quad . \tag{53}$$

As a matter of fact, the expression

$$\bar{\rho}(t) = \frac{\beta\mu t^{\beta-1}}{(1+\mu t^\beta)^2} \tag{54a}$$

is a fairly good approximation to ρ, even for values of β approaching unity. It has the correct behavior, for both large and small t, for $\beta = 1/2$, and for small t, with $\beta = 1$. One is tempted, in view of this, to consider β negative, either in (54a) or in (47b) and work out the consequences for the corresponding r(t); however, the result is simply a return, essentially, to the form (46); none of these expressions appear to be directly applicable to noise for which r(t) varies as a power of t greater negatively than -1. An even better expression, since it yields a more refined approximation to $\rho(t)$ for positive $\beta \leq 1$, and all t, is

$$\bar{\bar{\rho}}(t) = \frac{\beta\mu t^{\beta-1}}{\Gamma(1+\beta)}\left[1 + \frac{\mu t^\beta}{\Gamma(1+\beta)}\sqrt{\frac{\Pi\beta}{\sin\Pi\beta}}\right]^{-2} + \left(1 - \sqrt{\frac{\sin\Pi\beta}{\Pi\beta}}\right)\beta^n\mu e^{-\beta^n t} \; ; \tag{54b}$$

with an adjustable parameter, n. It is easy to see that $\int_0^\infty \bar{\bar{\rho}} dt = 1$. This may bear further investigation.

5.2. The Discrete Case

We now turn to the summation equation (9) and the subsidiary conditions (10) and (11).

a. An Iterative Method of Solution.

Let us put, for notational convenience

$$\left.\begin{aligned} r_i &= \alpha_{i+1} \\ \alpha_1 &= 1 \end{aligned}\right\} (i \geq 1). \tag{55}$$

Because of Equation (10) and $\alpha_1 = 1$, we can now rewrite (9) in either form

$$\alpha_i = \sum_{j=1}^{i} \rho_{i-j}\alpha_j + \delta_i^1 \tag{56a}$$

or

$$\alpha_i = \sum_{j=1}^{\infty} \rho_{i-j}\alpha_j + \delta_i^1 \ . \tag{56b}$$

Thus, given the ρ's, with $\rho_o = 0$, the rule for finding successive α's after $\alpha_1 = 1$ is obvious. Furthermore, we easily see, using (11), that

$$\sum_{i=1}^{\infty} \alpha_i = 1 + \sum_{j=1}^{\infty} r_j = \infty \ . \tag{57}$$

Equation (57) is clearly the analogue of Equation (21), just as (11) is the analogue of (8).

For convenience, we list the first few explicit formulas for the successive α's and r's:

$$\alpha_1 = 1$$

$$r_1 = \alpha_2 = \rho_1 \tag{58}$$

$$r_2 = \alpha_3 = \rho_1^2 + \rho_2$$

$$r_3 = \alpha_4 = \rho_1^3 + 2\rho_1\rho_2 + \rho_3$$

$$r_4 = \alpha_5 = \rho_1^4 + 3\rho_1^2\rho_2 + 2\rho_1\rho_3 + \rho_2^2 + \rho_4$$

$$r_5 = \alpha_6 = \rho_1^5 + 4\rho_1^3\rho_2 + 3\rho_1^2\rho_3 + 3\rho_1\rho_2^2 + 2\rho_1\rho_4 + 2\rho_2\rho_3 + \rho_5$$

$$r_6 = \alpha_7 = \rho_1^6 + 5\rho_1^4\rho_2 + 4\rho_1^3\rho_3 + 6\rho_1^2\rho_2^2 + 3\rho_1^2\rho_4 + 6\rho_1\rho_2\rho_3 + \rho_2^3 + 2\rho_1\rho_5 + $$
$$+ 2\rho_2\rho_4 + \rho_3^2 + \rho_6$$

The rule is:

"The n^{th} α is the sum of the products of the previous ones: α_{n-1}, α_{n-2}, $\cdots$, α_1, by ρ_1, ρ_2, $\cdots$, ρ_{n-1}, in that order; the number of terms is 2^{n-2} for $n \geq 2$."

Consequently, since all ρ's are positive, with a unit sum, they are also less than or equal to 1. Let ρ_{max} be the maximum ρ-value, and it must dominate any product of ρ's. Hence,

$$r_i = \alpha_{i+1} \leq 2^{i-1}\rho_{max}, \qquad i \geq 1 \tag{59}$$

and

$$\sum_{i=1}^{N} r_i \leq \rho_{max}(1+2+\cdots+2^{N-1}) = (2^N-1)\rho_{max}. \tag{60}$$

b. Alternative Solution Methods

There are two alternative methods for solving Equation (9) corresponding to the use of Laplace and Fourier transforms in the continuous case. Neither, however, appears to have any special computational advantage over the foregoing direct iterative approach.

For completeness, however, we describe them briefly.

(1) Power Series Method

Let us define $r_o = \rho_o = 0$, and

$$\left.\begin{aligned} \hat{r}(x) &= \sum_{n=o}^{\infty} r_n x^n \\ \hat{\rho}(x) &= \sum_{n=o}^{\infty} \rho_n x^n \end{aligned}\right\} \tag{61}$$

so that $\hat{\rho}(1) = 1$, in view of (11). $\hat{r}$ and $\hat{\rho}$ are generating functions for this discrete renewal process (Ref (10).

Then the summation equation (9) can be rewritten

$$r_n = \rho_n + \sum_{n'=o}^{\infty} \rho_{n-n'} r_{n'} \quad . \tag{62}$$

Transformations of the type (61) are easily inverted, e.g.:

$$r_n = \frac{1}{n!} \left[\frac{d^n}{dx^n} \hat{r}(x) \right]_{x=o} \quad . \tag{63}$$

Multiplication of (62) by x^n and adding leads easily to

$$\begin{aligned} \hat{r}(x) &= \hat{\rho}(x) + \sum_{n=o}^{\infty} \sum_{n'=o}^{\infty} \rho_{n-n'} x^{n-n'} r_{n'} x^{n'} \quad . \\ &= \hat{\rho}(x) + \hat{\rho}(x) \hat{r}(x) \quad . \end{aligned} \tag{64}$$

Hence

$$\hat{r}(x) = \frac{\hat{\rho}(x)}{1 - \hat{\rho}(x)} \tag{65a}$$

or

$$\hat{\rho}(x) = \frac{\hat{r}(x)}{1 + \hat{r}(x)} \quad . \tag{65b}$$

From (65a) and (63) we have

$$
\begin{aligned}
r_n &= \frac{1}{n!}\left[\frac{d^n}{dx^n}\,\frac{\hat{\rho}}{1-\hat{\rho}}\right]_{x=o} \\
&= \frac{1}{n!}\left[\frac{d^n}{dx^n}\,(\hat{\rho}+\hat{\rho}^2+\cdots)\right]_{x=o} \quad .
\end{aligned}
\tag{66}
$$

This gives the formal solution, and indeed, since

$$
\hat{\rho}^k = \sum_{n=o}^{\infty}\left(\sum_{\substack{n', n'', \cdots n^{(k)}: \\ n' + \cdots n^{(k)} = n}} \rho_{n'}\rho_{n''}\cdots\rho_{n^{(k)}}\right) x^n \quad , \tag{67}
$$

we see on examination that the formal solution is identical with the iterative solution, as given by (58) and the succeeding rule.

(2) Fourier Series Method

An alternative approach utilizes the orthogonality properties of the $(2N+1)^{th}$ roots of unity

$$
\omega_k = e^{2\pi i \frac{k}{2N+1}} \qquad (k = -N, \cdots 0, \cdots N) \tag{68}
$$

viz:

$$
\sum_{k=-N}^{N} e^{2\pi i (j-\ell)\frac{k}{2N+1}} = (2N+1)\delta^j_\ell \quad , \qquad \begin{pmatrix} -N \le j \le N \\ -N \le \ell \le N \end{pmatrix} \quad . \tag{69}
$$

Let

$$R_n = \begin{cases} \dfrac{r_n}{\sqrt{2N+1}} & , \ N \geq n > 0 \\ 0 & , \ -N \leq n \leq 0 \end{cases} \tag{70a}$$

$$P_n = \begin{cases} \dfrac{\rho_n}{\sqrt{2N+1}} & , \ N \geq n > 0 \\ 0 & , \ -N \leq n \leq 0 \end{cases} \tag{70b}$$

where N is a number sufficiently large, for computational purposes, so that all significant contributions to ρ_n (and r_n) are included in the specified range of n. With these definitions, the summation Equation (9) becomes

$$R_n = P_n + \frac{1}{\sqrt{2N+1}} \sum_{n'=-N}^{N} P_{n-n'} R_{n'}, \qquad (-N \leq n \leq N) \quad . \tag{71}$$

Define

$$\widetilde{R}_k = \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^{N} e^{\frac{2\pi ik}{2N+1}n} R_n, \qquad (-N \leq k \leq N) \tag{72}$$

and similarly for $\widetilde{P}_k$. Then, we easily see that Equation (71) can be transformed to

$$\widetilde{R}_k = \widetilde{P}_k + \widetilde{R}_k \widetilde{P}_k \quad . \tag{73}$$

The solution for r_n is then obviously

$$r_n = \frac{1}{2N+1} \sum_{n'=-N}^{N} e^{-\frac{2\pi i n n'}{2N+1}} \left[\frac{\sum_{n''=-N}^{N} P_{n''} e^{\frac{2\pi i n' n''}{2N+1}}}{1 + \frac{1}{\sqrt{2N+1}} \sum_{n''=-N}^{N} P_{n''} e^{2\pi i n'' \frac{n'}{2N+1}}} \right] \tag{74}$$

where $-N \leq n \leq N$.

6. CONCLUSIONS

From the foregoing analyses we conclude that the integral equation, relating the distribution of times between correct readings to the correct-reading rate, leads to an adequate theoretical description for white noise and random walk noise clock reading processes, summarized on pages 18, 19 and 20. It is indicative of the correct relationship for the limiting case of flicker noise rate, but is inadequate to describe lower order noises, i.e., $\beta < 0$, whose dispersion rates accelerate from a zero rate. It is likely that a more general approach, utilizing more fully the general probability distribution in phase space,

$$D_t(\tau, \dot{\tau}),$$

will be needed. On the other hand, recent investigations, utilizing the properties of a certain "normalization" function, show some promise of yielding definitive results for lower order noises. Perhaps the most important result of the present investigation is the recognition that the clock process considered is of standard renewal type.

The writer expresses his appreciation to Dr. J. Barnes for suggesting this problem, and to him and Dr. D. Halford for many useful and stimulating suggestions.

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8. APPENDIX I

Proof that $\int_0^\infty r(t)dt = \infty$. (Ref. p. 11, Equation (21))

From Equations (6) and (7) we see that

$$r(t) = \rho(t) + \int_0^\infty r(t')\rho(t-t')dt' \quad .$$

Integration over the infinite range of t yields (with $\tau = t - t'$), and using Equation (8),

$$\int_0^\infty r(t)dt = 1 + \int_0^\infty dt \int_0^\infty r(t')\rho(t-t')dt'$$

$$= 1 + \int_0^\infty r(t')dt' \int_{-t'}^\infty \rho(\tau)d\tau$$

$$= 1 + \int_0^\infty r(t')dt' \quad .$$

The required result follows immediately. Note also that for $t = 0$,

$$r(0) = \rho(0) \quad .$$

Furthermore, the very use of the convolution integral requires that $r(t')$ be integrable in the neighborhood of $t' = 0$.

9. APPENDIX II

The Laplace-Transform of Convolutions (Ref p. 11)

Consider the integral

$$C(t) = \int_0^t \rho(t-\tau)r(\tau)d\tau \quad .$$

In this, r(t) is not necessarily defined for negative t. However, because of condition (7), this can be rewritten

$$C(t) = \int_0^\infty \rho(t-\tau)r(\tau)d\tau$$

and hence also, if r(t) is defined to be zero for negative t,

$$C(t) = \int_0^\infty r(t-\tau)\rho(\tau)d\tau \quad .$$

These latter forms are most properly designated as convolution integrals. The Laplace-transform of C(t) is

$$\begin{aligned}
\widetilde{C}(\lambda) &= \int_0^\infty e^{-\lambda t}C(t)dt \\
&= \int_0^\infty\int_0^\infty e^{-\lambda(t-\tau)}\rho(t-\tau)e^{-\lambda\tau}r(\tau)d\tau dt \\
&= \int_0^\infty d\tau\int_{-\tau}^\infty e^{-\lambda t'}\rho(t')\,dt'r(\tau)d\tau \\
&= \widetilde{\rho}(\lambda)\widetilde{r}(\lambda) \qquad \text{, because of Equation (7).}
\end{aligned}$$

An alternative argument, leading to the same result, but not requiring the use of Condition (7), is as follows:

Let $\tau = ty$, in the expression for C(t). Then

$$C(t) = t\int_0^1 \rho(t[1-y])r(ty)dy \quad ,$$

and

$$\tilde{C}(\lambda) = \int_0^\infty e^{-\lambda t}\int_0^1 \rho(t[1-y]r(ty)t\,dy\,dt \quad .$$

Set

$$\xi = t - ty$$
$$\tau = ty$$

for $t > 0$, and $0 \le y \le 1$, and set $(\xi, \tau) = (0, 0)$ for $t = 0$, and $y = 1/2$.

Inversely

$$t = \xi + \tau$$
$$y = \frac{\tau}{\xi + \tau}$$

for $(\xi, \tau) \neq (0, 0)$, and $(t, y) = (0, 1/2)$ for $\xi = 0$ and $\tau = 0$. By this mapping the region

$$\begin{cases} 0 \le y \le 1 \\ 0 < t \le \infty \end{cases}$$

and $\quad y = 1/2, \; t = 0 ,$

in the (y, t)-plane corresponds to the region

$$0 \le \xi \le \infty$$
$$0 \le \tau \le \infty$$

in the (ξ, τ)-plane, in a one-one fashion. Moreover, in the domain of continuous mapping

$$d\xi\, d\tau = t\, dy\, dt \quad .$$

Hence, unless ρ or r have quite exceptional behaviors near t = 0, we find that

$$\widetilde{C}(\lambda) = \int_0^\infty \int_0^\infty e^{-\lambda(\xi + \tau)} \rho(\xi) r(\tau) d\xi \, d\tau$$

$$= \widetilde{\rho}(\lambda) \widetilde{r}(\lambda)$$

as before.

10. APPENDIX III

The Laplace Inversion Integral for $\rho(t)$ (Ref p. 12, Equations (27), (28); p. 18)

If we substitute the expresion for $\tilde{r}(\lambda)$ from Equation (28) into the inversion integral of Equation (27), and allow λ to be a complex variable z whose argument is restricted to its principal value

$$-\pi < \arg z \leq \pi,$$

then we are faced with the problem of evaluating

$$\rho(t) = \frac{\mu}{2\pi i} \lim_{\delta \to \infty} \int_{\gamma - i\delta}^{\gamma + i\delta} e^{zt} \frac{dz}{\mu + z^{\beta}}, \text{ for } \beta > 0.$$

The zeros of the denominator in the integrand occur at

$$z = z_n^{\pm} = \mu^{(1/\beta)} e^{\pm i\pi(2n+1)/\beta}$$

for n = 0, 1, 2, $\cdots$. But the restriction on the argument of z means that only those n-values (or poles) need be considered for which

$$2n + 1 \leq \beta.$$

Two cases arise, which will be considered separately:

(A) $\beta = 2m + 1$, an odd integer, and

(B) β is not an odd integer.

Case (A) $\beta = 2m + 1$.

The number of poles of the integrand is 2m + 1, since for n = m, only the real zero at

$$z_m^{+} = \mu^{1/\beta} e^{i\pi} = -\mu^{1/\beta}$$

lies in the z-plane as defined, while z_m^- does not. For the other poles, $0 \le n \le \frac{\beta - 1}{2} = m$.

Choose a rectangular contour C consisting of

(1) a line segment L^+, parallel to the imaginary axis on which

$$z = \gamma + iy, \quad -\delta \le y \le \delta$$

(2) a line segment M^+, parallel to the real axis on which

$$z = x + i\delta, \quad \gamma \ge x \ge -\gamma'$$

(3) a line segment L^-, parallel to the imaginary axis on which

$$z = -\gamma' + iy, \quad +\delta \ge y \ge -\delta$$

and,

(4) a line segment M^-, parallel to the real axis on which

$$z = x - i\delta, \quad -\gamma' \le x \le \gamma$$

where $(\gamma, \gamma', \delta)$ are all positive and large enough so that C contains all the poles of the integrand in its interior. We know that the contour integral

$$\frac{1}{2\pi i} \oint_C e^{zt} \frac{dz}{\mu + z^\beta}$$

equals the sum of the residues of the integrand at its (2m+1) poles. Now, the integrals:

$$\int_{M^\pm} = \mp e^{\pm i\delta t} \int_0^{\gamma} e^{xt} \frac{dx}{\mu + (x \pm i\delta)^\beta} \pm e^{\pm i\delta t} \int_0^{-\gamma'} e^{xt} \frac{dx}{\mu + (x \pm i\delta)^\beta}$$

$$\int_{L^-} = \int_{\delta}^{-\delta} e^{-\gamma' t} e^{iyt} \frac{i\,dy}{\mu + (-\gamma' + iy)^\beta}$$

all clearly vanish, as δ, γ' approach ∞. Hence, we have, in this limit

$$\rho(t) = \mu \sum \text{ residues at the } (2m+1) \text{-poles.}$$

The residue at $z_n^{\pm}$ is

$$\exp(t\mu^{1/\beta} e^{\pm i\pi(2n+1)/\beta} \Big/ \beta \mu^{(\beta-1)/\beta} e^{\pm i\pi(2n+1)(\beta-1)/\beta}$$

so in this case

$$\rho(t) = \frac{\mu^{1/\beta}}{\beta}\left[e^{-\mu^{(1/\beta)}t} + \sum_{n=0}^{\frac{\beta-3}{2}} 2e^{\mu^{(1/\beta)}t\cos\pi(2n+1)/\beta} \cos\left\{\mu^{1/\beta} t \sin(\pi(2n+1)/\beta) + -\pi(2n+1)(\beta-1)/\beta\right\}\right].$$

For $\beta = 1$, we have the "white noise situation",

$$\rho = \mu e^{-\mu t}.$$

<u>Case (B).</u> <u>$(2m - 1 < \beta < 2m + 1)$</u>

The number of poles of the integrand is $2m$, where $(m - 1)$ is the largest value of n for which $2n + 1 < \beta$. Clearly $0 \le n \le m - 1$.

Let us introduce a cut along the negative real axis of the z-plane, even though the integrand is continuous across this cut when β is a positive even integer.

The new integration contour, C', consists of the previous rectangle (except that L^- is cut at $y = 0$) plus the following:

(1) two line segments, $Q^{\pm}$ parallel to the x-axis, on which, for sufficiently small $\epsilon > 0$,

$$z = \sigma e^{\pm \pi} \pm i\epsilon$$

where

$$\gamma' \le \sigma \le a,$$

and

(2) a circular contour, A, of radius "a" around the origin (cut at the negative axis) on which

$$z = ae^{i\theta}$$

and
$$-\pi + \eta \leq \theta \leq \pi - \eta$$

where η is chosen so as to connect the circle with $Q^{\pm}$. Again, the contour integral

$$\frac{1}{2\pi i} \oint_{C'} e^{zt} \frac{dz}{\mu + z^{\beta}}$$

equals the sum of the polar residues.

The integral:

$$\int_A = \int_{-\pi+\eta}^{\pi-\eta} \exp(ae^{i\theta}t) \frac{aie^{i\theta}d\theta}{\mu + a^{\beta}e^{i\beta\theta}}$$

clearly vanishes as $a \to 0$.

The integrals along $Q^{\pm}$ are:

$$\int_{Q^+} + \int_{Q^-} = \int_{\gamma'}^{a} e^{[i\epsilon - \sigma]t} \frac{e^{+i\pi}d\sigma}{\mu + [\sigma e^{i\pi} + i\epsilon]^{\beta}} + \int_{a}^{\gamma'} e^{[i\epsilon - \sigma]t} \frac{e^{-i\pi}d\sigma}{\mu + [\sigma e^{-i\pi} - i\epsilon]^{\beta}} .$$

As ϵ approaches zero, these integrals become equal to

$$\int_{a}^{\gamma'} e^{-\sigma t} d\sigma \left[\frac{1}{\mu + \sigma^{\beta}e^{+i\pi\beta}} - \frac{1}{\mu + \sigma^{\beta}e^{-i\pi\beta}} \right]$$

and the integral along L^- has its former value.

Let γ' and δ become infinite as before, and let "a" approach zero. Then, in this limit

$$\rho(t) = \mu \sum (\text{residues at the 2m-poles}) + I(t)$$

where

$$I(t) = \frac{\mu}{2\pi i} \int_0^\infty e^{-\sigma t} d\sigma \left[\frac{1}{\mu + \sigma^\beta e^{-i\pi\beta}} - \frac{1}{\mu + \sigma^\beta e^{i\pi\beta}} \right] .$$

If we set $\nu = \sigma t$, combine the split terms, and factor out μ^2, we easily find

$$I(t) = \frac{\sin \pi\beta}{\pi \mu t^{\beta+1}} \int_0^\infty e^{-\nu} d\nu \frac{\nu^\beta}{1 + \frac{2 \cos \pi\beta}{\mu}\left(\frac{\nu}{t}\right)^\beta + \frac{1}{\mu^2}\left(\frac{\nu}{t}\right)^{2\beta}}$$

as in the text, Equation (47a), while a simple rearrangement of factors in the split terms leads immediately, with this substitution, to Equation (47b). In this we assume that $0 < \beta < 1$, so that the residue sum does not contribute. When $\beta = 1$, we get the white noise form for $\rho(t)$, Equation (47c), and if $\beta > 1$ and not an odd integer, we have the formula

$$\rho(t) = 2\frac{\mu^{1/\beta}}{\beta} \sum_{n=0}^{m-1} e^{\mu^{1/\beta} t \cos\left(\pi \frac{2n+1}{\beta}\right)} \cos \left\{ \mu^{1/\beta} t \sin\left(\pi \frac{2n+1}{\beta}\right) - \pi \frac{\beta - 1}{\beta}(2n+1) \right\} + I(t) .$$

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